

# Computing topological zeta functions of groups, algebras, and modules, II

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Building on our previous work [22], we develop the first practical algorithm for computing topological zeta functions of nilpotent groups, non-associative algebras, and modules. While we previously depended upon non-degeneracy assumptions, the theory developed here allows us to overcome these restrictions in various interesting cases, far extending the scope of [22].

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## 1 Introduction

**Topological zeta functions.** A recent addition to the theory of zeta functions of algebraic structures, topological zeta functions of groups and algebras were introduced by du Sautoy and Loeser [12] as asymptotic invariants related to the enumeration of subobjects. They are limits as the “prime tends to one” of the local subobject zeta functions due to

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Grunewald, Segal, and Smith [15] in the same way that topological zeta functions of polynomials due to Denef and Loeser [9] are limits of Igusa’s local zeta functions [17].

For an informal explanation of this limit, recall that for a finitely generated torsion-free nilpotent group  $G$ , the local subgroup zeta function  $\zeta_{G,p}(s)$  of  $G$  at the prime  $p$  is given by the Dirichlet series  $\zeta_{G,p}(s) = \sum_{e=0}^{\infty} a_{p^e}(G)p^{-es}$ , where  $a_{p^e}(G)$  denotes the number of subgroups of index  $p^e$  in  $G$ . Informally, the topological subgroup zeta function  $\zeta_{G,\text{top}}(s)$  of  $G$  is the rational function in  $s$  obtained as the constant term of  $(1 - p^{-1})^d \zeta_{G,p}(s)$  as a series in  $p - 1$ , where  $d$  is the Hirsch length of  $G$ . For example, it is well-known that  $\zeta_{\mathbf{Z}^d,p}(s) = \frac{1}{(1-p^{-s})(1-p^{1-s})\dots(1-p^{d-1-s})}$  and we find that  $\zeta_{\mathbf{Z}^d,\text{top}}(s) = \frac{1}{s(s-1)\dots(s-(d-1))}$ .

A decade after their introduction, apart from a short list of examples in [12, §9], topological zeta functions of groups, algebras, and modules remained uncharted territory. It is the purpose of the project begun in [22] and continued here to change that.

**Central objects: toric data.** At the heart of the present article lies the notion of a *toric datum*. A toric datum consists of a half-open cone within some Euclidean space and a finite collection of Laurent polynomials. We will begin our study of toric data in §3, where we will also relate them to the “cone integral data” of du Sautoy and Grunewald [11]. As we will see, toric data give rise to associated  $p$ -adic integrals (closely related to the “cone integrals” from [11]) and to topological zeta functions by means of a limit “ $p \rightarrow 1$ ”. Most importantly, topological zeta functions arising from the enumeration of subgroups, subalgebras, and submodules can be expressed in terms of toric data.

In general, the computation of such zeta functions relies on the same impractical ingredient as the computation of cone integrals: resolution of singularities. In suitably non-degenerate settings, explicit resolutions can be obtained via so-called “toric modifications” [21]. Based on such classical results from toric geometry and previous applications to Igusa-type zeta functions ([8], in particular), [22] provides us with explicit convex-geometric formulae for topological zeta functions associated with toric data under non-degeneracy assumptions. Unfortunately, when it comes to the computation of topological subgroup, subalgebra, or submodule zeta functions, the practical scope of [22] on its own is limited: more often than not, the non-degeneracy assumptions are violated for examples of interest.

**Main result.** Our main result, Algorithm 4.1, is a practical algorithm which seeks to compute topological zeta functions associated with toric data in favourable situations, significantly extending the applicable range of the ideas in [22]. The practicality of Algorithm 4.1 is demonstrated by a computer implementation [23] which also includes a database containing a substantial number of topological subalgebra and ideal zeta functions computed using Algorithm 4.1. These computations provide strong evidence for the intriguing features of such topological zeta functions predicted by the conjectures in [22, §8].

Algorithm 4.1 is based on a series of algebraic and convex-geometric operations, a subset of which constitutes an algorithmic version of [22, Thm 6.7]. By adding further steps, we extend the scope of Algorithm 4.1 beyond the non-degeneracy assumptions of [22]. Some of these steps (such as *balancing* in §5.2) can be interpreted within the

geometric framework of toric modifications. Others (e.g. *simplification* in §7.2) are most naturally regarded as systematic and generalised versions of “tricks” for the evaluation of  $p$ -adic integrals previously employed by Woodward [30] in a semi-automatic fashion. Yet others (such as *reduction* in §7.3) are inspired by the theory of Gröbner bases.

**Previous computations.** While Algorithm 4.1 is the first of its kind specifically designed to compute the *topological* zeta functions considered here, a significant number of *local* zeta functions of groups and algebras have been previously computed, see e.g. [13, 29]. Although the informal definition of topological zeta functions from above suggests that they can be deduced from local formulae, a rigorous approach requires additional information, cf. [12, §§7.9.3] and see [22, Rem. 5.20].

A significant proportion of the known local zeta functions of groups and algebras were found by Woodward using a combination of machine-computations and human insight [30]. The number of zeta functions he managed to compute is particularly impressive in view of the relatively elementary nature of his method which is based on skillful (but ultimately ad hoc) applications of certain “tricks” [30, §2.3.2] for computing with  $p$ -adic integrals. Unfortunately, due to the reliance of his computations on human guidance, they are difficult to reproduce.

Although, as we mentioned before, some of Woodward’s “tricks” can be regarded as special cases of the simplification step in §7.2, the  $p$ -adic part of our method does not generalise his approach. In particular, using Algorithm 4.1, we managed to determine topological zeta functions whose local versions Woodward could not compute (see §9.1) and, conversely, there are examples of local zeta functions computed by him whose topological counterparts cannot be determined using Algorithm 4.1.

The techniques developed in the present article are specifically designed for the computation of topological zeta functions of groups, algebras, and modules. However, through our use of [22], we draw heavily upon formulae for Igusa-type zeta functions and associated topological zeta functions that have been obtained under non-degeneracy assumptions, see, in particular, [8, 9, 28].

**Outline.** After a brief reminder on local zeta functions of groups, algebras, and modules in §2, we introduce toric data and associated  $p$ -adic integrals in §3. The central section of the present article is §4 which is devoted to describing our main algorithm and its various components; details on the latter will be provided in subsequent sections. In §5, we consider toric data which are balanced and regular—these two notions provide the main link between the present article and its predecessor [22]. In particular, as we will explain in §6, topological zeta functions associated with regular toric data can be computed using [22]. In order to keep the present article reasonably self-contained, key facts from [22] will be briefly recalled as needed. The two remaining ingredients of Algorithm 4.1, namely simplification and reduction are discussed in §7. Practical aspects and the author’s implementation *Zeta* [23] of Algorithm 4.1 are briefly discussed in §8. Finally, in §9, we consider specific examples which illustrate key steps of Algorithm 4.1 and which also demonstrate its practical strength.

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## Notation

The symbol “ $\subset$ ” signifies not necessarily proper inclusion. We let  $\mathbf{N}$ ,  $\mathbf{Z}$ ,  $\mathbf{R}$ , and  $\mathbf{C}$  denote the natural numbers (without zero), integers, real and complex numbers, respectively. We write  $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$ . By a  $p$ -adic field, we mean a finite extension of the field  $\mathbf{Q}_p$  of  $p$ -adic numbers. Throughout this article,  $k$  is a number field with ring of integers  $\mathfrak{o}$ . We let  $K$  denote a  $p$ -adic field endowed with an embedding  $K \supset k$ . We write  $\mathfrak{O}_K$  and  $\mathfrak{P}_K$  for the valuation ring of  $K$  and its maximal ideal, respectively. We further let  $\pi_K$  denote a uniformiser and let  $\nu_K$  be the valuation on  $K$  with  $\nu_K(\pi) = 1$ ; we write  $\nu_K(\mathbf{x}) = (\nu_K(x_1), \dots, \nu_K(x_n))$  for  $\mathbf{x} = (x_1, \dots, x_n) \in K^n$ . Further write  $|x|_K = q_K^{-\nu_K(x)}$  and  $\|M\|_K = \sup(|x|_K : x \in M)$ , where  $q_K = |\mathfrak{O}_K/\mathfrak{P}_K|$  and  $M \subset K$ . Finally,  $\mu_K$  denotes the Haar measure on  $K^n$  with  $\mu_K(\mathfrak{O}_K^n) = 1$ , where  $n$  will be clear from the context. By a non-associative algebra, we mean a not necessarily associative one. Write  $\mathbf{T}^n = \text{Spec}(\mathbf{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$ . For a commutative ring  $R$ , we identify  $\mathbf{T}^n(R) = (R^\times)^n$  and write  $\mathbf{T}_R^n = \mathbf{T}^n \times \text{Spec}(R)$ . We often write  $\mathbf{X} = (X_1, \dots, X_n)$  and  $\mathbf{X}^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ .

## 2 Background: zeta functions of groups, algebras, and modules

The following is an abridged version of [22, §2]. We investigate the following mild generalisations of subring, ideal, and submodule zeta functions, cf. [15, 24].

**Definition 2.1.** Let  $R$  be the ring of integers in a number field or in a  $p$ -adic field.

- (i) Let  $M$  be a free  $R$ -module of finite rank and let  $\mathcal{E}$  be a subalgebra of  $\text{End}_R(M)$ . The **submodule zeta function** of  $\mathcal{E}$  acting on  $M$  is

$$\zeta_{\mathcal{E} \curvearrowright M}(s) = \sum_{n=1}^{\infty} \#\{U : U \text{ is an } \mathcal{E}\text{-submodule of } M \text{ with } |M : U| = n\} \cdot n^{-s}.$$

- (ii) Let  $\mathcal{A}$  be a non-associative  $R$ -algebra whose underlying  $R$ -module is free of finite rank. The **subalgebra zeta function** of  $\mathcal{A}$  is

$$\zeta_{\mathcal{A}}(s) = \sum_{n=1}^{\infty} \#\{\mathcal{U} : \mathcal{U} \text{ is an } R\text{-subalgebra of } \mathcal{A} \text{ with } |\mathcal{A} : \mathcal{U}| = n\} \cdot n^{-s}.$$

Let  $\Omega(\mathcal{A})$  be the  $\mathfrak{o}$ -subalgebra of  $\text{End}_{\mathfrak{o}}(\mathcal{A})$  generated by  $x \mapsto ax$  and  $x \mapsto xa$  with  $a$  ranging over  $\mathcal{A}$ . Then the **ideal zeta function** of  $\mathcal{A}$  is  $\zeta_{\mathcal{A}}^{\triangleleft}(s) = \zeta_{\Omega(\mathcal{A}) \curvearrowright \mathcal{A}}(s)$ .

When  $\mathfrak{o} = \mathbf{Z}$ , we refer to non-associative  $\mathbf{Z}$ -algebras as non-associative rings, to subalgebras as subrings, etc. Let  $\text{Tr}_d(R)$  be the ring of upper triangular  $d \times d$ -matrices over  $R$ . The following result of du Sautoy and Grunewald is stated in the version from [22].

**Theorem 2.2** ([11, §5]).

- (i) Let  $\mathcal{A}$  be a non-associative  $\mathfrak{o}$ -algebra which is free of rank  $d$  as an  $\mathfrak{o}$ -module. Then there exists a finite set  $\mathbf{f} \subset \mathfrak{o}[X_{ij}^{\pm 1} : 1 \leq i \leq j \leq d]$  with the following property: if  $K \supset k$  is a  $p$ -adic field, then

$$\zeta_{\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{D}_K}(s) = (1 - q_K^{-1})^{-d} \int_{\{x \in \mathrm{Tr}_d(\mathfrak{D}_K) : \|\mathbf{f}(x)\|_K \leq 1\}} |x_{11}|_K^{s-1} \cdots |x_{dd}|_K^{s-d} d\mu_K(x), \quad (2.1)$$

where we identified  $\mathrm{Tr}_d(K) \approx K^{\binom{d+1}{2}}$  and we regarded  $\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{D}_K$  as an  $\mathfrak{D}_K$ -algebra.

- (ii) Let  $\mathcal{E}$  be an  $\mathfrak{o}$ -subalgebra of  $\mathrm{End}_{\mathfrak{o}}(M)$ , where  $M$  is a free  $\mathfrak{o}$ -module of rank  $d$ . Then there are Laurent polynomials as in (i) such that the conclusion of (i) holds for  $\zeta_{(\mathcal{E} \otimes_{\mathfrak{o}} \mathfrak{D}_K) \curvearrowright (M \otimes_{\mathfrak{o}} \mathfrak{D}_K)}(s)$  in place of  $\zeta_{\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{D}_K}(s)$ .

As for subgroups, the enumeration of normal subgroups of a torsion-free finitely generated nilpotent group  $G$  gives rise to local normal subgroup zeta functions  $\zeta_{G,p}^{\mathfrak{A}}(s)$ .

**Theorem 2.3** ([15, §4]). Let  $G$  be a finitely generated torsion-free nilpotent group with associated Lie  $\mathbf{Q}$ -algebra  $\mathfrak{L}(G)$  under the Mal'cev correspondence. Let  $\mathcal{L} \subset \mathfrak{L}(G)$  be a  $\mathbf{Z}$ -subalgebra which is finitely generated as a  $\mathbf{Z}$ -module and whose  $\mathbf{Q}$ -span is  $\mathfrak{L}(G)$ . Then for almost all primes  $p$ , we have  $\zeta_{G,p}(s) = \zeta_{\mathcal{L} \otimes_{\mathbf{Z}} \mathbf{Z}_p}(s)$  and  $\zeta_{G,p}^{\mathfrak{A}}(s) = \zeta_{\mathcal{L} \otimes_{\mathbf{Z}} \mathbf{Z}_p}^{\mathfrak{A}}(s)$ .

**Constructing Laurent polynomials from algebras and modules.** We now recall the explicit description of  $\mathbf{f}$  in Theorem 2.2 given by du Sautoy and Grunewald; our exposition is equivalent to [22, Rem. 2.7(ii)]. First, choose  $\mathfrak{o}$ -bases of  $\mathcal{A}$  or  $M$  in Theorem 2.2 to identify  $\mathcal{A} = \mathfrak{o}^d$  or  $M = \mathfrak{o}^d$  as  $\mathfrak{o}$ -modules, respectively. We are then either given a bilinear multiplication  $\beta: \mathfrak{o}^d \otimes_{\mathfrak{o}} \mathfrak{o}^d \rightarrow \mathfrak{o}^d$  turning  $\mathfrak{o}^d$  into an  $\mathfrak{o}$ -algebra or a finite generating set  $\mathcal{M}$  of  $\mathcal{E} \subset \mathrm{M}_d(\mathfrak{o})$ . Let  $R := \mathfrak{o}[X_{ij} : 1 \leq i \leq j \leq d]$  and let  $C := [X_{ij}]_{i \leq j} \in \mathrm{Tr}_d(R)$  with rows  $C_1, \dots, C_d$ . We think of  $C$  as parameterising a generic  $\mathfrak{o}$ -submodule of  $\mathfrak{o}^d$  via its row span. We extend  $\beta$  to a map  $R^d \otimes_R R^d \rightarrow R^d$  in the natural way. For Theorem 2.2(i), let  $\mathbf{f}$  consist of the non-zero entries of  $\det(C)^{-1} \beta(C_m, C_n) \mathrm{adj}(C)$  for  $1 \leq m, n \leq d$ ; for part (ii), we instead consider the entries of  $\det(C)^{-1} (CM) \mathrm{adj}(C)$  as  $M$  ranges over  $\mathcal{M}$ .

### 3 Toric data

In this section, we introduce the basic object for all of our algorithms: toric data. These objects are closely related to the cone integral data introduced in [11]. In particular, they also give rise to associated  $p$ -adic integrals and topological zeta functions.

#### 3.1 Basics

By a **half-open cone** in  $\mathbf{R}^n$  we mean a set of the form

$$\mathcal{C}_0 = \{\omega \in \mathbf{R}^n : \langle \phi_1, \omega \rangle, \dots, \langle \phi_d, \omega \rangle \geq 0, \quad \langle \chi_1, \omega \rangle, \dots, \langle \chi_e, \omega \rangle > 0\},$$

where  $\langle -, - \rangle$  denotes the standard inner product and  $\phi_i, \chi_j \in \mathbf{R}^n$ . We say that  $\mathcal{C}_0$  is **rational** if we may choose the  $\phi_i, \chi_j$  among elements of  $\mathbf{Z}^n$ .

**Definition 3.1.** A **toric datum** in  $n$  variables over  $k$  is a pair  $\mathcal{T} = (\mathcal{C}_0; \mathbf{f})$  consisting of a half-open rational cone  $\mathcal{C}_0 \subset \mathbf{R}_{\geq 0}^n$  and a finite family  $\mathbf{f} = (f_1, \dots, f_r)$  of Laurent polynomials  $f_1, \dots, f_r \in k[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ .

We often write  $(\mathcal{C}_0; f_1, \dots, f_r)$  instead of  $(\mathcal{C}_0; \mathbf{f})$ . Furthermore, we usually omit the references to  $n$  and  $k$ . The non-negativity assumption  $\mathcal{C}_0 \subset \mathbf{R}_{\geq 0}^n$  is included to ensure the convergence of the integrals in §3.2 below. As we will explain in Remark 3.4, toric data provide us with a convenient formalism for describing the domain of integration for the integrals in Theorem 2.2. Recall the notational conventions from p. 4.

**Notation 3.2.** Given  $\mathcal{T} = (\mathcal{C}_0; \mathbf{f})$  as in Definition 3.1 and a  $p$ -adic field  $K \supset k$ , we write

$$\mathcal{T}_K := \left\{ \mathbf{x} \in \mathbf{T}^n(K) : \nu_K(\mathbf{x}) \in \mathcal{C}_0, \|\mathbf{f}(\mathbf{x})\|_K \leq 1 \right\}.$$

**Toric data and cone conditions.** For an explanation of our terminology, suppose that  $\mathcal{C}_0$  is closed. Then each of the conditions  $\nu_K(\mathbf{x}) \in \mathcal{C}_0$  and  $\|\mathbf{f}(\mathbf{x})\|_K \leq 1$  can be expressed as a conjunction of finitely many divisibility conditions  $v(\mathbf{x}) \mid w(\mathbf{x})$ , where  $v, w \in k[\mathbf{X}]$ . Indeed, let  $\mathcal{C}_0 = \{\omega \in \mathbf{R}^n : \langle \phi_1, \omega \rangle, \dots, \langle \phi_d, \omega \rangle \geq 0\}$  for  $\phi_1, \dots, \phi_d \in \mathbf{Z}^n$  and  $\mathbf{f} = (f_1, \dots, f_r)$ . It is easy to see that  $(\mathcal{C}_0; \mathbf{f})_K = (\mathbf{R}_{\geq 0}^n; f_1, \dots, f_r, \mathbf{X}^{\phi_1}, \dots, \mathbf{X}^{\phi_d})_K$ . Next, for a Laurent polynomial  $g = \mathbf{X}^{-\gamma} g^+$  with  $\gamma \in \mathbf{N}_0^n$  and  $g^+ \in \mathfrak{O}_K[\mathbf{X}]$ , the condition  $|g(\mathbf{x})|_K \leq 1$  (where  $\mathbf{x} \in \mathbf{T}^n(K) \cap \mathfrak{O}_K^n = (\mathbf{R}_{\geq 0}^n; \emptyset)_K$ ) is equivalent to  $\mathbf{x}^\gamma \mid g^+(\mathbf{x})$ .

Hence, for  $\mathcal{C}_0$  closed, a toric datum gives rise to a special case of a “cone condition” as defined in [11, Def. 1.2(1)]. Moreover, if  $\mathcal{D}_K$  denotes the set of  $K$ -points of such a cone condition, then  $(\mathcal{C}_0; \mathbf{f})_K = \mathcal{D}_K \cap \mathbf{T}^n(K)$ .

**Definition 3.3.** A toric datum  $(\mathcal{C}_0; \mathbf{f})$  is **trivial** if  $\mathcal{C}_0 = \emptyset$ .

### 3.2 Zeta functions associated with toric data

Let  $\beta \in M_{m \times n}(\mathbf{N}_0)$  with rows  $\beta_1, \dots, \beta_m$ . Given a toric datum  $\mathcal{T}$  in  $n$  variables over  $k$  and a  $p$ -adic field  $K \supset k$ , we consider the “zeta function” defined by

$$Z_K^{\mathcal{T}, \beta}(s_1, \dots, s_m) := \int_{\mathcal{T}_K} |\mathbf{x}^{\beta_1}|_K^{s_1} \cdots |\mathbf{x}^{\beta_m}|_K^{s_m} d\mu_K(\mathbf{x}), \quad (3.1)$$

where  $s_1, \dots, s_m \in \mathbf{C}$  with  $\operatorname{Re}(s_j) \geq 0$ ; convergence is guaranteed by the non-negativity assumptions  $\mathcal{C}_0 \subset \mathbf{R}_{\geq 0}^n$ ,  $\beta_1, \dots, \beta_m \in \mathbf{N}_0^n$ . We note that  $Z_K^{\mathcal{T}, \beta}(s_1, \dots, s_m)$  is a special case of the zeta functions studied in [22, §4].

**Remark 3.4** (Local subalgebra and submodule zeta functions). Disregarding factors of the form  $(1 - q_K^{-1})^{\pm d}$ , Theorem 2.2 shows that zeta functions associated with toric data generalise local zeta functions arising from the enumeration of (normal) subgroups, subalgebras, and submodules as in §2, cf. [22, Rem. 4.12]. Indeed, let  $\mathcal{A}$  (or  $M$ ) in

Theorem 2.2 have rank  $d$  and let  $n = d(d+1)/2$ . We identify  $\mathrm{Tr}_d \approx \mathbf{A}^n$  via  $(x_{ij}) \mapsto (x_{11}, \dots, x_{1d}, x_{22}, \dots, x_{dd})$  and let  $\mathbf{f} \subset \mathfrak{o}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be an associated family of Laurent polynomials as in Theorem 2.2; see the end of §2 for an explicit construction. Let  $\mathcal{T} = (\mathbf{R}_{\geq 0}^n; \mathbf{f})$  and let  $\beta \in \mathrm{M}_{d \times n}(\mathbf{N}_0)$  be the matrix whose  $j$ th row corresponds to the elementary matrix with entry 1 in position  $(j, j)$  under the above isomorphism  $\mathrm{Tr}_d \approx \mathbf{A}^n$ . Then  $Z_K^{\mathcal{T}, \beta}(s_1, \dots, s_m)$  specialises to the integral in Theorem 2.2 via  $(s_1, \dots, s_m) \mapsto (s-1, \dots, s-d)$ .

**Relationship with cone integrals.** In §3.1, we explained how toric data give rise to cone conditions from [11] (at least when  $\mathcal{C}_0$  is closed). In the same spirit, we may regard the integrals in (3.1) as special cases of (multivariate versions of) the cone integrals in [11]; the pair  $(\mathcal{T}, \beta)$  takes the place of the “cone integral data” in [11]. One of the two important special features of (3.1) compared with cone integrals is that we insist on left-hand sides in the divisibility conditions describing the domain of integration being monomial (just as they are in Theorem 2.2). By expressing such divisibility conditions in terms of Laurent polynomials as in §3.1, we naturally adopt a “toric” point of view. This perspective will prove to be especially useful in combination with the second key feature of the integrals (3.1), namely the presence of a not necessarily closed ambient half-open cone  $\mathcal{C}_0$ . Focusing exclusively on integrals of the shape (3.1) allows us to develop specialised techniques for manipulating and evaluating them.

**Evaluation in theory: “explicit formulae”.** Consider the  $K$ -indexed family of zeta functions  $Z_K^{\mathcal{T}, \beta}$  defined in terms of a toric datum  $\mathcal{T}$  and a matrix  $\beta$  in (3.1). Going back to work of Denef [7] and du Sautoy and Grunewald [10], using powerful but typically impractical techniques such as resolution of singularities, it can be shown (cf. [22, Ex. 5.11(vi)]) that there are finitely many  $W_i(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) \in \mathbf{Q}(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  and  $k$ -varieties  $V_i$  for  $i \in I$ , say, with the following property: if  $K \supset k$  is a  $p$ -adic field, then, unless  $\mathfrak{p}_K := \mathfrak{o} \cap \mathfrak{P}_K$  belongs to some finite exceptional set (depending on  $\mathcal{T}$  only),

$$Z_K^{\mathcal{T}, \beta}(s_1, \dots, s_m) = \sum_{i \in I} \# \bar{V}_i(\mathfrak{O}_K / \mathfrak{P}_K) \cdot W_i(q_K, q_K^{-s_1}, \dots, q_K^{-s_m}), \quad (3.2)$$

where  $\bar{V}_i$  is the reduction modulo  $\mathfrak{p}_K$  of  $V_i$ . We understand the task of “computing” the  $Z_K^{\mathcal{T}, \beta}(s_1, \dots, s_m)$  to be the explicit construction of  $V_i$  and  $W_i(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  as in (3.2).

**Topological zeta functions.** As originally observed by Denef and Loeser [9] for Igusa’s local zeta function, given an “explicit formula” as in (3.2), under additional assumptions regarding the shapes of the  $W_i(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  (see §6.1), we may “pass to the limit  $q_K \rightarrow 1$ ” and obtain the associated topological zeta function

$$Z_{\mathrm{top}}^{\mathcal{T}, \beta}(s_1, \dots, s_m) = \sum_{i \in I} \chi(V_i(\mathbf{C})) \cdot [W_i](s_1, \dots, s_m) \in \mathbf{Q}(s_1, \dots, s_m); \quad (3.3)$$

here  $\chi(V_i(\mathbf{C}))$  denotes the topological Euler characteristic with respect to any embedding of  $k$  into  $\mathbf{C}$  and  $[W_i](s_1, \dots, s_m) \in \mathbf{Q}(s_1, \dots, s_m)$  is the constant term of

$W_i(q_K, q_K^{-s_1}, \dots, q_K^{-s_m})$ , formally expanded as a series in  $q_K - 1$ . A particularly noteworthy consequence of [9] is that the right-hand side of (3.3) is independent of the choice of the family  $(V_i, W_i(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m))_{i \in I}$  in (3.2).

**Evaluation in practice: non-degeneracy.** Let  $\mathcal{T} = (\mathcal{C}_0; \mathbf{f})$  be a toric datum over  $k$ . If  $\mathbf{f}$  is non-degenerate relative to  $\mathcal{C}_0$  in the sense of [22, Def. 4.2(i)], then [22, Thm 4.10] yields an effective version of (3.2) in the sense that it provides *explicit* descriptions of varieties  $V_i$  and rational functions  $W_i(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  in terms of convex-geometric data associated with various cones and polytopes attached to  $\mathcal{T}$  and  $\beta$ , see Theorem 5.8. While further computations involving these objects might still be expensive or even infeasible in large dimensions, if available, they are however much more useful than formulae obtained using general resolution algorithms; the latter are usually only practical for  $n \leq 3$ . Recall from Remark 3.4 that for the computation of subalgebra or submodule zeta functions,  $n = d(d+1)/2$ , where  $d$  is the additive rank of the object under consideration.

If  $\mathbf{f}$  is even globally non-degenerate (see [22, Def. 4.2(ii)]), then the Euler characteristics in (3.3) can be expressed in terms of mixed volumes via the Bernstein-Kushnirenko-Khovanskii Theorem, yielding an effective form of (3.3), see [22, Thm 6.7]. The theory underpinning Algorithm 4.1 in §4 (to be developed in the present article) draws upon and extends this result to overcome certain instances of degeneracy.

**Topological subalgebra and submodule zeta functions.** Under rather weak technical assumptions on the  $W_i(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  appearing in (3.3), passing from local to topological zeta functions commutes with affine specialisations of the variables  $s_1, \dots, s_m$ , see [22, Rem. 5.15]. In particular, we obtain univariate versions of (3.2) and (3.3) arising from the integrals in Theorem 2.2 and thus rigorous definitions of topological subalgebra and submodule zeta functions, see [22, Def. 5.17] or §6.1 below.

## 4 The main algorithm

In this section, we give a high-level description of an algorithm which seeks to compute topological zeta functions associated with toric data. The main application that we have in mind is the computation of topological subalgebra and submodule zeta functions via the univariate specialisations explained in Remark 3.4.

**The algorithm.** We suppose that we are given a toric datum  $\mathcal{T}^0 = (\mathcal{C}_0; \mathbf{f})$  in  $n$  variables over  $k$ —in practice, we are primarily interested in the case where  $\mathbf{f}$  is a family of Laurent polynomials arising from Theorem 2.2 and  $\mathcal{C}_0 = \mathbf{R}_{\geq 0}^n$  (where  $n = d(d+1)/2$ ) as in Remark 3.4. Given  $\mathcal{T}^0$  and  $\beta \in M_{m \times n}(\mathbf{N}_0)$ , the function `TOPOLOGICALZETAFUNCTION` (Algorithm 4.1) attempts to compute the topological zeta function  $Z_{\text{top}}^{\mathcal{T}^0, \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m) \in \mathbf{Q}(\mathbf{s}_1, \dots, \mathbf{s}_m)$  associated with the integrals  $Z_K^{\mathcal{T}^0, \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m)$  in (3.1); see §6.2 for a rigorous definition of  $Z_{\text{top}}^{\mathcal{T}^0, \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m)$ . We note that from now on, we use bold face letters  $\mathbf{s}_j$  to distinguish variables over  $\mathbf{Q}$  from the complex numbers  $s_j$  in §3.2.



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**Algorithm 4.1** TOPOLOGICALZETAFUNCTION( $\mathcal{T}^0, \beta$ )

---

**Input:** a toric datum  $\mathcal{T}^0$  in  $n$  variables over  $k$ , a matrix  $\beta \in M_{m \times n}(\mathbf{N}_0)$

**Output:** the topological zeta function  $Z_{\text{top}}^{\mathcal{T}^0, \beta}(s_1, \dots, s_m) \in \mathbf{Q}(s_1, \dots, s_m)$  or **fail**

```

1: unprocessed  $\leftarrow [\mathcal{T}^0]$ , regular  $\leftarrow []$ , ▷ Stage I
2: while unprocessed is non-empty do
3:   remove an element  $\mathcal{T}$  from unprocessed
4:    $\mathcal{T} \leftarrow \text{SIMPLIFY}(\mathcal{T})$ 
5:   if  $\mathcal{T}$  is not balanced then
6:     new  $\leftarrow \text{BALANCE}(\mathcal{T})$ 
7:   else if  $\mathcal{T}$  is regular then
8:     add  $\mathcal{T}$  to regular
9:     new  $\leftarrow []$ 
10:  else
11:    new  $\leftarrow \text{REDUCE}(\mathcal{T})$ 
12:    if new = fail then return fail
13:    add the non-trivial elements of new to unprocessed
14: return  $\sum_{\mathcal{T} \in \text{regular}} \text{EVALUATETOPOLOGICALLY}(\mathcal{T}, \beta)$  ▷ Stage II

```

---

We now explain the structure of Algorithm 4.1 and the roles played by the functions SIMPLIFY, BALANCE, REDUCE, and EVALUATETOPOLOGICALLY. Details will be given in the following sections.

**Stage I: the main loop.** During the first stage (lines 1–13) of Algorithm 4.1, we maintain two lists, **unprocessed** and **regular**, of toric data. The essential point here is that unless the execution of Algorithm 4.1 is aborted in line 12, each iteration of the loop in lines 2–13 preserves the following property:

- (♣) There exists a finite  $S \subset \text{Spec}(\mathfrak{o})$  such that if  $K \supset k$  is a  $p$ -adic field with  $\mathfrak{o} \cap \mathfrak{P}_K \notin S$ , then  $\mathcal{T}_K^0$  is the disjoint union of all  $\mathcal{T}_K$  with  $\mathcal{T}$  ranging over **unprocessed**  $\sqcup$  **regular**.

In particular, we always have  $Z_{\text{top}}^{\mathcal{T}^0, \beta}(s_1, \dots, s_m) = \sum_{\mathcal{T}} Z_{\text{top}}^{\mathcal{T}, \beta}(s_1, \dots, s_m)$ , where  $\mathcal{T}$  again ranges over **unprocessed**  $\sqcup$  **regular** (Lemma 6.7).

The central notions, to be defined in §5, featuring in the while-loop in Algorithm 4.1 are those of *balanced* and *regular* toric data. The function BALANCE (see §5.2) takes as input an arbitrary toric datum and returns a distinguished family of associated balanced ones. Among balanced toric data, regular ones constitute a subclass which is intimately related to the concept of non-degeneracy used in [22]. Namely, given a toric datum  $\mathcal{T} = (\mathcal{C}_0; \mathbf{f})$ , the family  $\mathbf{f}$  is non-degenerate relative to  $\mathcal{C}_0$  in the sense of [22, Def. 4.2(i)] if and only if each element of  $\text{BALANCE}(\mathcal{T})$  is regular (Proposition 5.7). Our objective during the first stage of Algorithm 4.1 is to successively modify and decompose toric data until, hopefully, at some point all toric data under consideration will be regular. In the reduction step (lines 11–12), given a balanced toric datum  $\mathcal{T}$  that fails to be regular, the function REDUCE (see §7.3) attempts to mitigate this failure of regularity by allowing us to replace  $\mathcal{T}$  by a suitable family of toric data. This step is supported by the function SIMPLIFY (see §7.2) which removes certain redundancies from its input.

**Stage II: topological evaluation.** After successful termination of the while-loop in Algorithm 4.1, the computation of the topological zeta function  $Z_{\text{top}}^{\mathcal{T}^0, \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m)$  is reduced to computing the topological zeta functions associated with a (possibly large) number of regular toric data via the function `EVALUATETOPOLOGICALLY` (see §6.7). Given a regular toric datum, by the aforementioned connection between regularity and non-degeneracy and as previously indicated in §3.2, [22] provides us with explicit convex-geometric  $p$ -adic formulae. Using [22, §6], we may then deduce a formula for  $Z_{\text{top}}^{\mathcal{T}^0, \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m)$  in terms of the topological Euler characteristics of certain subvarieties of tori. However, in contrast to the assumption of global non-degeneracy in [22, Thm 6.7], regularity alone does not suffice to provide us with explicit formulae for these Euler characteristics.

**Remark 4.1.**

- (i) Write  $(\mathcal{D}_0; \mathbf{g}) = \text{SIMPLIFY}(\mathcal{T}^0)$ . If  $\mathbf{g}$  is non-degenerate relative to  $\mathcal{D}_0$  in the sense of [22, Def. 4.2(i)], then the first stage of Algorithm 4.1 will always succeed (see Remark 6.15(ii)). If  $\mathbf{g}$  is even globally non-degenerate (see [22, Def. 4.2(ii)]), then `TOPOLOGICALZETAFUNCTION` constitutes an algorithmic version of [22, Thm 6.7] for the class of topological zeta functions considered here. However, Algorithm 4.1 can do much more: the reduction (lines 11–12) and simplification (line 4) steps allow it to overcome certain instances of degeneracy. When it comes to the practical computation of topological subalgebra and submodule zeta functions, the computations documented in §9 demonstrate that Algorithm 4.1 substantially extends the scope of [22]. It is however important to note that a possible point of failure remains in Algorithm 4.1, namely as part of the reduction step in line 12.
- (ii) The separation of Algorithm 4.1 into two stages might seem artificial. It is justified by the aforementioned chance of failure of Algorithm 4.1 and the observations that the final evaluation in line 14 is often the most computationally expensive step.
- (iii) The first stage of Algorithm 4.1 is  $p$ -adic in nature in the sense that it consists entirely of manipulations of  $p$ -adic integrals. In contrast, the second stage is inherently topological. Finding a practically useful  $p$ -adic version of the second stage and thus a practical method for computing associated local zeta functions is a natural direction for future research.

## 5 Balanced and regular toric data

### 5.1 Background: cones, polytopes, and polynomials

The following summary of well-known material is based upon [22, §§3.1, 3.3, 4.1].

**Cones.** A (closed) **cone** in  $\mathbf{R}^n$  is a set of the form  $\mathcal{C} = \{\omega \in \mathbf{R}^n : \langle \phi_1, \omega \rangle, \dots, \langle \phi_d, \omega \rangle \geq 0\}$ , where  $\phi_1, \dots, \phi_d \in \mathbf{R}^n$ . Equivalently, cones in  $\mathbf{R}^n$  are sets of the form

$$\text{cone}(\varrho_1, \dots, \varrho_r) := \mathbf{R}_{\geq 0} \cdot \varrho_1 + \dots + \mathbf{R}_{\geq 0} \cdot \varrho_r,$$

where  $\varrho_1, \dots, \varrho_r \in \mathbf{R}^n$ ; we call the  $\varrho_1, \dots, \varrho_r$  a system of **generators** of  $\text{cone}(\varrho_1, \dots, \varrho_r)$ . A half-open cone as defined in §3.1 is precisely a set of the form  $\mathcal{C} \setminus (\mathcal{C}_1 \cup \dots \cup \mathcal{C}_r)$ , where  $\mathcal{C}$  is a cone,  $r \geq 0$ , and  $\mathcal{C}_1, \dots, \mathcal{C}_r$  are faces of  $\mathcal{C}$ . In particular, cones are half-open cones.

**Polytopes and normal cones.** We insist that faces of a polytope be non-empty. For a non-empty polytope  $\mathcal{P} \subset \mathbf{R}^n$  and  $\omega \in \mathbf{R}^n$ , let  $\text{face}_\omega(\mathcal{P})$  denote the face of  $\mathcal{P}$  where  $\langle -, \omega \rangle$  attains its minimum. If  $\mathcal{Q} \subset \mathbf{R}^n$  is another non-empty polytope, then  $\text{face}_\omega(\mathcal{P} + \mathcal{Q}) = \text{face}_\omega(\mathcal{P}) + \text{face}_\omega(\mathcal{Q})$ . The **normal cone** of a face  $\tau \subseteq \mathcal{P}$  is the relatively open cone  $N_\tau(\mathcal{P}) = \{\omega \in \mathbf{R}^n : \text{face}_\omega(\mathcal{P}) = \tau\}$  of dimension  $n - \dim(\tau)$ . The  $N_\tau(\mathcal{P})$  partition  $\mathbf{R}^n$  into relatively open cones. The normal cones of faces of  $\mathcal{P} + \mathcal{Q}$  coincide with the non-empty sets of the form  $N_\tau(\mathcal{P}) \cap N_v(\mathcal{Q})$  for faces  $\tau \subset \mathcal{P}$  and  $v \subset \mathcal{Q}$ , corresponding to the unique decomposition of a face of  $\mathcal{P} + \mathcal{Q}$  into a sum of the form  $\tau + v$ .

**Newton polytopes and initial forms.** Let  $f = \sum_{\alpha \in \mathbf{Z}^n} c_\alpha \mathbf{X}^\alpha \in k[\mathbf{X}^{\pm 1}]$ , where  $c_\alpha \in k$ . The **support** of  $f$  is  $\text{supp}(f) = \{\alpha \in \mathbf{Z}^n : c_\alpha \neq 0\}$ . The **Newton polytope**  $\text{New}(f)$  of  $f$  is the convex hull of  $\text{supp}(f)$  within  $\mathbf{R}^n$ . For  $f, g \in k[\mathbf{X}^{\pm 1}]$ , we have  $\text{New}(fg) = \text{New}(f) + \text{New}(g)$ . For  $\omega \in \mathbf{R}^n$ , the **initial form**  $\text{in}_\omega(f)$  of  $f$  in the direction  $\omega$  is the sum of those  $c_\alpha \mathbf{X}^\alpha$  with  $\alpha \in \text{supp}(f)$  where  $\langle \alpha, \omega \rangle$  attains its minimum. If  $f \neq 0$ , then  $\text{face}_\omega(\text{New}(f)) = \text{New}(\text{in}_\omega(f))$  for  $\omega \in \mathbf{R}^n$ . The equivalence classes on  $\mathbf{R}^n$  of  $\omega \sim \omega' : \iff \text{in}_\omega(f) = \text{in}_{\omega'}(f)$  are the normal cones of the faces of  $\text{New}(f)$ .

## 5.2 Balanced toric data

### Definition 5.1.

- (i) Let  $\mathcal{C}_0 \subset \mathbf{R}^n$  be an arbitrary subset. We say that  $f \in k[\mathbf{X}^{\pm 1}]$  is  **$\mathcal{C}_0$ -balanced** if  $\text{in}_\omega(f)$  is constant as  $\omega$  ranges over  $\mathcal{C}_0$ . If  $\mathcal{C}_0 \neq \emptyset$ , we denote this common initial form by  $\text{in}_{\mathcal{C}_0}(f)$ .
- (ii) A toric datum  $(\mathcal{C}_0; f_1, \dots, f_r)$  is **balanced** if each  $f_i$  is  $\mathcal{C}_0$ -balanced.

**Example 5.2.** Let  $n = 2$  and  $k$  be arbitrary. Define  $f_1 = X_1^{-1} - X_2^{-1}$  and  $f_2 = X_1^{-2} - X_2^{-2}$ . Then  $(\mathbf{R}_{\geq 0}^2; f_1, f_2)$  is not balanced. For instance,  $\text{in}_{(1,0)}(f_1) = X_1^{-1}$  but  $\text{in}_{(0,1)}(f_1) = -X_2^{-1}$ . Let  $\mathcal{C}_0 = \{(\omega_1, \omega_2) \in \mathbf{R}_{\geq 0}^2 : \omega_1 > \omega_2\}$ . Then  $(\mathcal{C}_0; f_1, f_2)$  is balanced with  $\text{in}_{\mathcal{C}_0}(f_1) = X_1^{-1}$  and  $\text{in}_{\mathcal{C}_0}(f_2) = X_1^{-2}$ .

**Lemma 5.3.** Let  $\mathcal{T} = (\mathcal{C}_0; f_1, \dots, f_r)$  be a toric datum with  $f_1 \cdots f_r \neq 0$ . Let  $\mathcal{N} = \text{New}(f_1 \cdots f_r)$ . Then  $\mathcal{T}$  is balanced if and only if  $\mathcal{C}_0 \subset N_\tau(\mathcal{N})$  for some face  $\tau \subset \mathcal{N}$ .

*Proof.* We may assume that  $\mathcal{C}_0 \neq \emptyset$ . Suppose that  $\mathcal{T}$  is balanced. Then, for  $1 \leq i \leq r$ , since  $f_i$  is  $\mathcal{C}_0$ -balanced, there exists a face  $\tau_i \subset \text{New}(f_i)$  with  $\mathcal{C}_0 \subset N_{\tau_i}(\text{New}(f_i))$ . Hence,  $\emptyset \neq \mathcal{C}_0 \subset \bigcap_{i=1}^r N_{\tau_i}(\text{New}(f_i))$  and  $\tau := \sum_{i=1}^r \tau_i$  is the desired face of  $\sum_{i=1}^r \text{New}(f_i) = \mathcal{N}$ . Conversely, let  $\mathcal{C}_0 \subset N_\tau(\mathcal{N})$  and write  $\tau = \sum_{i=1}^r \tau_i$  for faces  $\tau_i \subset \text{New}(f_i)$ . Then  $\mathcal{C}_0 \subset N_\tau(\mathcal{N}) \subset N_{\tau_i}(\text{New}(f_i))$  for  $1 \leq i \leq r$ , whence  $\mathcal{T}$  is balanced.  $\blacklozenge$

The following notion will be used to show that Algorithm 4.1 preserves ( $\clubsuit$ ) from p. 9.

**Definition 5.4.** Let  $\mathcal{T}$  be a toric datum. A finite collection  $\mathfrak{C}$  of toric data is a **partition** of  $\mathcal{T}$  if there exists a finite  $S \subset \text{Spec}(\mathfrak{o})$  such that if  $K \supset k$  is a  $p$ -adic field with  $\mathfrak{o} \cap \mathfrak{P}_K \notin S$ , then  $\mathcal{T}_K = \bigcup_{\mathcal{T}' \in \mathfrak{C}} \mathcal{T}'_K$  (disjoint).

We may thus rephrase ( $\clubsuit$ ) by stating that **unprocessed**  $\sqcup$  **regular** is a partition of  $\mathcal{T}^0$ .

**The function BALANCE.** Given a toric datum  $(\mathcal{C}_0; f_1, \dots, f_r)$ , the function BALANCE produces a partition consisting of balanced toric data as follows. Let  $I = \{i : f_i \neq 0\}$  and  $\mathcal{N} = \text{New}(\prod_{i \in I} f_i)$ . Then let  $\text{BALANCE}(\mathcal{C}_0; f_1, \dots, f_r)$  return the collection of  $(\mathcal{C}_0 \cap N_\tau(\mathcal{N}); f_1, \dots, f_r)$  for faces  $\tau \subset \mathcal{N}$  with  $\mathcal{C}_0 \cap N_\tau(\mathcal{N}) \neq \emptyset$ . Note that each toric datum  $(\mathcal{C}_0 \cap N_\tau(\mathcal{N}); f_1, \dots, f_r)$  is balanced by Lemma 5.3.

### 5.3 Regular toric data

Let  $\bar{k}$  be an algebraic closure of  $k$ .

**Definition 5.5.** We say that a balanced toric datum  $(\mathcal{C}_0; f_1, \dots, f_r)$  is **regular** if either  $\mathcal{C}_0 = \emptyset$  or the following condition is satisfied:

For all  $J \subset \{1, \dots, r\}$ , if  $\mathbf{u} \in \mathbf{T}^n(\bar{k})$  satisfies  $\text{in}_{\mathcal{C}_0}(f_j)(\mathbf{u}) = 0$  for all  $j \in J$ , then the Jacobian matrix  $\left[ \frac{\partial \text{in}_{\mathcal{C}_0}(f_j)}{\partial X_i}(\mathbf{u}) \right]_{i=1, \dots, n; j \in J}$  has rank  $|J|$ .

We say that  $(\mathcal{C}_0; f_1, \dots, f_r)$  is **singular** if it is balanced but not regular.

**Example 5.6.** The toric datum  $(\mathcal{C}_0; f_1, f_2)$  in Example 5.2 is trivially regular since both initial forms  $\text{in}_{\mathcal{C}_0}(f_1)$  and  $\text{in}_{\mathcal{C}_0}(f_2)$  are Laurent monomials and hence do not vanish on  $\mathbf{T}^2(\bar{k})$ . Let  $\mathcal{C}'_0 = \{(\omega_1, \omega_2) \in \mathbf{R}_{\geq 0}^2 : \omega_1 = \omega_2\}$ . Then  $(\mathcal{C}'_0; f_1, f_2)$  is balanced but singular. Indeed, the initial forms are  $\text{in}_{\mathcal{C}'_0}(f_1) = f_1$  and  $\text{in}_{\mathcal{C}'_0}(f_2) = f_2$  and the condition in Definition 5.5 is violated on the subvariety of  $\mathbf{T}_k^2$  defined by  $X_1 = X_2$  for  $J = \{1, 2\}$ .

By definition, a balanced toric datum  $(\mathcal{C}_0; \mathbf{f})$  with  $0 \notin \mathbf{f}$  is regular if and only if  $\mathbf{f}$  is non-degenerate relative to  $\mathcal{C}_0$  in the sense of [22, Def. 4.2(i)]. More generally, the following holds by construction.

**Lemma 5.7.** *Given  $(\mathcal{C}_0; \mathbf{f})$  with  $0 \notin \mathbf{f}$ , the family  $\mathbf{f}$  is non-degenerate relative to  $\mathcal{C}_0$  in the sense of [22, Def. 4.2(i)] if and only if each element of  $\text{BALANCE}(\mathcal{C}_0; \mathbf{f})$  is regular.  $\blacklozenge$*

**Testing regularity.** As a part of Algorithm 4.1, we need to test regularity of toric data. This can be carried out using Gröbner bases computations as follows. Let  $(\mathcal{C}_0; f_1, \dots, f_r)$  be a balanced toric datum with  $\mathcal{C}_0 \neq \emptyset$ . Write  $g_i = \text{in}_{\mathcal{C}_0}(f_i)$  and  $M_J = \left[ \frac{\partial g_j}{\partial X_i} \right]_{i=1, \dots, n; j \in J}$ . By the weak Nullstellensatz,  $(\mathcal{C}_0; \mathbf{f})$  is regular if and only if for each  $J \subset \{1, \dots, r\}$ , the Laurent polynomials  $g_j$  for  $j \in J$  together with the  $|J| \times |J|$ -minors of  $M_J$  generate the unit ideal of  $k[\mathbf{X}^{\pm 1}]$ . For practical computations, it is convenient to rephrase the latter condition in terms of the polynomial algebra  $k[\mathbf{X}]$ . Thus, since regularity of  $(\mathcal{C}_0; f_1, \dots, f_r)$  is invariant under rescaling of the  $f_i$  by Laurent monomials (cf. [22, Rem. 4.3(ii)]), we

may assume that  $g_1, \dots, g_r \in k[\mathbf{X}]$  are polynomials. It follows that  $(\mathcal{C}_0; f_1, \dots, f_r)$  is regular if and only if for all  $J$ , the monomial  $X_1 \cdots X_n$  is contained in the radical of the ideal generated by all  $g_j$  ( $j \in J$ ) and the  $|J| \times |J|$ -minors of  $M_J$  within  $k[\mathbf{X}]$ . Using the Rabinowitsch trick, the latter condition can now be tested using Gröbner bases machinery.

## 5.4 Reminder: generating functions of cones

The following material is largely well-known, see e.g. [4, Ch. 13] and [25, §4.5].

**Generating functions.** Given a cone  $\mathcal{C} \subset \mathbf{R}^n$  and a ring  $R$ , let  $R[\mathcal{C}_0 \cap \mathbf{Z}^n]$  be the  $R$ -subalgebra of  $R[\mathbf{X}^{\pm 1}]$  spanned by  $\mathbf{X}^\alpha$  with  $\alpha \in \mathcal{C} \cap \mathbf{Z}^n$ . If  $\mathcal{C} \subset \mathbf{R}_{\geq 0}^n$  is a *rational* cone, then, within the field of fractions of  $\mathbf{Q}[\mathbf{X}]$ , the series  $\sum_{\omega \in \mathcal{C} \cap \mathbf{N}_0^n} \mathbf{X}^\omega \in \mathbf{Q}[\mathbf{X}]$  is given by a rational function  $|\mathcal{C}|$  of the form  $|\mathcal{C}| = f(\mathbf{X}) / \prod_{i=1}^r (1 - \mathbf{X}^{\alpha_i})$ , where  $f(\mathbf{X}) \in \mathbf{Z}[\mathcal{C} \cap \mathbf{Z}^n]$  and  $\alpha_1, \dots, \alpha_r \in \mathcal{C} \cap \mathbf{N}_0^n$ . For an analytic characterisation of  $|\mathcal{C}|$ , let  $\mathcal{C} = \text{cone}(\varrho_1, \dots, \varrho_e)$  for  $0 \neq \varrho_j \in \mathbf{N}_0^n$ . Then  $U(\mathcal{C}) := \{\mathbf{x} \in \mathbf{T}^n(\mathbf{C}) : |\mathbf{x}^{\varrho_j}| < 1 \text{ for } j = 1, \dots, e\}$  is a non-empty open set which is independent of the choice of  $\varrho_1, \dots, \varrho_e$ , and we have  $|\mathcal{C}|(\mathbf{x}) = \sum_{\omega \in \mathcal{C} \cap \mathbf{N}_0^n} \mathbf{x}^\omega$  for all  $\mathbf{x} \in U(\mathcal{C})$ , the convergence being absolute and compact on  $U(\mathcal{C})$ .

**Triangulation.** The function  $|\mathcal{C}|$  can be computed in terms of a triangulation of  $\mathcal{C}$ . Here, by a **triangulation** of  $\mathcal{C}$ , we mean a rational polyhedral fan  $\mathcal{F}$  in  $\mathbf{R}^n$  which consists of simplicial cones and whose support is  $\mathcal{C}$ . By the inclusion-exclusion principle, we may write  $|\mathcal{C}|$  as a  $\mathbf{Z}$ -linear combination of the rational functions  $|\sigma|$  for  $\sigma \in \mathcal{F}$ ; those  $\sigma \in \mathcal{F}$  with  $\dim(\sigma) = \dim(\mathcal{C})$  have coefficient 1. If  $\sigma$  is simplicial, say  $\sigma = \text{cone}(\beta_1, \dots, \beta_d)$  with  $d = \dim(\sigma)$ , then  $|\sigma| = (\sum \mathbf{X}^\alpha) / \prod_{i=1}^d (1 - \mathbf{X}^{\beta_i})$ , where the summation in the numerator extends over the lattice points in the half-open parallelepiped  $\{\sum_{i=1}^d a_i \beta_i : a_i \in \mathbf{R}, 0 \leq a_i < 1\}$ . Using the inclusion-exclusion principle once again, what has been said about closed cones above extends to (rational) half-open cones  $\mathcal{C}_0 \subset \mathbf{R}_{\geq 0}^n$ , see [22, §3.1]; in particular, we obtain a rational function  $|\mathcal{C}_0|$  enumerating the lattice points in  $\mathcal{C}_0$ .

## 5.5 Local zeta functions associated with regular toric data

We record how the machinery developed in [22] provides convex-geometric formulae for local zeta functions associated with regular toric data.

**Monomial substitutions.** (Cf. [22, §3.2].) Let  $A \in M_{n \times (m+1)}(\mathbf{N}_0)$ . We assume that the first column of  $A$  is  $(1, \dots, 1)^\top$ . Write  $A_1, \dots, A_n$  for the rows of  $A$ . Let  $\mathbf{X} = (X_1, \dots, X_n)$  and  $\mathbf{Y} = (Y_0, \dots, Y_m)$  consist of independent variables over  $\mathbf{Q}$ . Let  $\mathcal{A}$  be the  $\mathbf{Q}$ -subalgebra of  $\mathbf{Q}(\mathbf{X})$  generated by  $\mathbf{Q}[\mathbf{X}]$  and all  $(1 - \mathbf{X}^\alpha)^{-1}$  for  $0 \neq \alpha \in \mathbf{N}_0^n$ ; similarly, let  $\mathcal{B}$  be the  $\mathbf{Q}$ -algebra generated by  $\mathbf{Q}[\mathbf{Y}]$  and all  $(1 - \mathbf{Y}^\beta)^{-1}$  for  $0 \neq \beta \in \mathbf{N}_0^{m+1}$ . Then  $\mathbf{X}^\alpha \mapsto \mathbf{Y}^{\alpha A}$  extends to a homomorphism  $(-)^A : \mathcal{A} \rightarrow \mathcal{B}$ . In particular, if  $\mathcal{C}_0 \subset \mathbf{R}_{\geq 0}^n$  is a rational half-open cone, then  $|\mathcal{C}_0|$  belongs to  $\mathcal{A}$  and we may thus consider its image  $|\mathcal{C}_0|^A$ . Observe that the rational function  $|\mathcal{C}_0|^A$  can be evaluated at any point  $(y_0, \dots, y_m) \in \mathbf{C}^{m+1}$  with  $0 \leq |y_0| < 1$  and  $0 \leq |y_j| \leq 1$  for  $j = 1, \dots, m$ .

**Setup.** Let  $(\mathcal{C}_0; f_1, \dots, f_r)$  be a regular toric datum over  $k$ . We assume that  $\mathcal{C}_0 \neq \emptyset$ . For  $1 \leq i \leq r$ , we choose an arbitrary element  $\gamma_i \in \text{supp}(\text{in}_{\mathcal{C}_0}(f_i))$ . Let  $J \subset \{1, \dots, r\}$ . Define  $V_J^\circ$  to be the subvariety of  $\mathbf{T}_k^n$  defined by the vanishing of all  $\text{in}_{\mathcal{C}_0}(f_j)$  for  $j \in J$  and the non-vanishing of all remaining  $\text{in}_{\mathcal{C}_0}(f_i)$ . For  $j \in J$ , let  $\delta_{jJ}$  be the  $j$ th standard basis vector of  $\mathbf{R}^J$  while for  $1 \leq i \leq r$ ,  $i \notin J$ , we let  $\delta_{iJ} = 0_{\mathbf{R}^J}$ . Define  $\mathcal{D}_J$  to be the cone consisting of those  $(\xi, o) \in \mathbf{R}_{\geq 0}^n \times \mathbf{R}_{\geq 0}^J$  with  $\langle \gamma_i, \xi \rangle + \langle \delta_{iJ}, o \rangle \geq 0$  for  $i = 1, \dots, r$ . Let  $\mathcal{C}_0^J := (\mathcal{C}_0 \times \mathbf{R}_{\geq 0}^J) \cap \mathcal{D}_J$ . Finally, for an  $m \times n$  matrix  $\beta$  with rows  $\beta_1, \dots, \beta_m$ , define an  $(n + |J|) \times (m + 1)$  matrix  $A_J(\beta) = [(1, \dots, 1)^\top, (\beta_1, 0)^\top, \dots, (\beta_m, 0)^\top]$ .

**Theorem 5.8** (Cf. [22, Thm 4.10]). *Let  $\mathcal{T} = (\mathcal{C}_0; f_1, \dots, f_r)$  be a regular toric datum over  $k$ . Let  $\beta \in M_{m \times n}(\mathbf{N}_0)$ . Define  $V_J^\circ$ ,  $\mathcal{C}_0^J$ , and  $A_J(\beta)$  as above. Then for all  $p$ -adic fields  $K \supset k$ , unless  $\mathfrak{p}_K = \mathfrak{o} \cap \mathfrak{P}_K$  belongs to some finite exceptional set (depending only on  $\mathcal{T}$ ), we have*

$$Z_K^{\mathcal{T}, \beta}(s_1, \dots, s_m) = \sum_{J \subset \{1, \dots, r\}} \# \bar{V}_J^\circ(\mathfrak{O}_K / \mathfrak{P}_K) \cdot \frac{(q_K - 1)^{|J|}}{q_K^n} \cdot |\mathcal{C}_0^J|^{A_J(\beta)}(q_K^{-1}, q_K^{-s_1}, \dots, q_K^{-s_m})$$

for  $s_1, \dots, s_m \in \mathbf{C}$  with  $\text{Re}(s_j) \geq 0$ , where  $\bar{\cdot}$  denotes reduction modulo  $\mathfrak{p}_K$ .

## 6 Topological zeta functions and regular toric data

We describe the function `EVALUATETOPologically` (see §6.7) which computes the topological zeta function  $Z_{\text{top}}^{\mathcal{T}, \beta}(s_1, \dots, s_m)$  associated with a regular toric datum  $\mathcal{T}$  and a matrix  $\beta$  as defined in §6.2. Our method is based on refined and algorithmic versions of the key ingredients of [22, Thm 6.7] applied to Theorem 5.8, with further extensions removing the assumptions of “global non-degeneracy” from [22, §6].

### 6.1 Reminder: topological zeta functions via $p$ -adic formulae

In §3.2, we sketched how formulae of the form (3.2) can be used to read off the associated topological zeta function (3.3). For a rigorous treatment, we now recall the formalism of [22] which is based on work of Denef and Loeser [9]. In particular, we recall the technical conditions regarding the  $W_i$  in (3.3) alluded to above.

**Formal binomial expansions.** Let  $\mathbf{q}, t_1, \dots, t_m, s_1, \dots, s_m$  be algebraically independent over  $\mathbf{Q}$ ; we regard these variables as symbolic versions of  $q_K, q_K^{-s_1}, \dots, q_K^{-s_m}, s_1, \dots, s_m$  in Theorem 5.8. Using the binomial series, we define  $\mathbf{q}^{-s_j} := \sum_{d=0}^{\infty} \binom{-s_j}{d} (\mathbf{q} - 1)^d \in \mathbf{Q}[s_j][[\mathbf{q} - 1]]$ . Let  $W(\mathbf{q}, t_1, \dots, t_m) \in \mathbf{Q}(\mathbf{q}, t_1, \dots, t_m)$  be of the form

$$W(\mathbf{q}, t_1, \dots, t_m) = \frac{f(\mathbf{q}, t_1, \dots, t_m)}{\prod_{i=1}^r (q^{a_i} t^{b_i} - 1)}, \quad (6.1)$$

where  $f(\mathbf{q}, t_1, \dots, t_m) \in \mathbf{Q}[\mathbf{q}^{\pm 1}, t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ ,  $(a_i, b_i) \in \mathbf{Z}^{1+m}$ ,  $(a_i, b_i) \neq (0, 0)$ . Given  $W(\mathbf{q}, t_1, \dots, t_m)$ , we obtain  $W(\mathbf{q}, \mathbf{q}^{-s_1}, \dots, \mathbf{q}^{-s_m}) \in \mathbf{Q}(s_1, \dots, s_m)((\mathbf{q} - 1))$ .

**Notation 6.1.**

- (i) Let  $\mathbf{M}$  be the  $\mathbf{Q}$ -subalgebra of  $\mathbf{Q}(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  consisting of those  $W(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  of the form (6.1) with  $W(\mathbf{q}, \mathbf{q}^{-s_1}, \dots, \mathbf{q}^{-s_m}) \in \mathbf{Q}(\mathbf{s}_1, \dots, \mathbf{s}_m)[\mathbf{q} - 1]$ .
- (ii) Given  $W(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) \in \mathbf{M}$ , we let  $[W](\mathbf{s}_1, \dots, \mathbf{s}_m) \in \mathbf{Q}(\mathbf{s}_1, \dots, \mathbf{s}_m)$  denote the constant term of  $W(\mathbf{q}, \mathbf{q}^{-s_1}, \dots, \mathbf{q}^{-s_m})$  as a series in  $\mathbf{q} - 1$ .

The following generalises topological zeta functions of polynomials introduced in [9].

**Theorem & Definition 6.2** ([22, §5.3]). *Let  $Z = (Z_K)$  be a family of rational functions  $Z_K(\mathbf{t}_1, \dots, \mathbf{t}_m) \in \mathbf{Q}(\mathbf{t}_1, \dots, \mathbf{t}_m)$  indexed by  $p$ -adic fields  $K \supset k$  (up to  $k$ -isomorphism). Suppose that there exists a finite family of  $k$ -varieties  $V_i$  and  $W_i(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) \in \mathbf{M}$  ( $i \in I$ ) such that for  $p$ -adic fields  $K \supset k$ , unless  $\mathfrak{p}_K = \mathfrak{o} \cap \mathfrak{P}_K$  belongs to some finite set,*

$$Z_K(\mathbf{t}_1, \dots, \mathbf{t}_m) = \sum_{i \in I} \# \bar{V}_i(\mathfrak{O}_K / \mathfrak{P}_K) \cdot W_i(q_K, \mathbf{t}_1, \dots, \mathbf{t}_m),$$

where  $\bar{\cdot}$  denotes reduction modulo  $\mathfrak{p}_K$ . Then the **topological zeta function**

$$Z_{\text{top}}(\mathbf{s}_1, \dots, \mathbf{s}_m) := \sum_{i \in I} \chi(V_i(\mathbf{C})) \cdot [W_i](\mathbf{s}_1, \dots, \mathbf{s}_m) \in \mathbf{Q}(\mathbf{s}_1, \dots, \mathbf{s}_m)$$

associated with  $Z$  is independent of the choice of  $(V_i, W_i(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m))_{i \in I}$ .

When  $m = 1$ , we write  $\mathbf{s}$  and  $\mathbf{t}$  instead of  $\mathbf{s}_1$  and  $\mathbf{t}_1$ . The formulae for local subalgebra and submodule zeta functions in [11] give rise to associated topological zeta functions.

**Theorem & Definition 6.3** ([22, §5.4]).

- (i) *Let  $\mathcal{A}$  be a non-associative  $\mathfrak{o}$ -algebra whose underlying  $\mathfrak{o}$ -module is free of finite rank  $d$ . For a  $p$ -adic field  $K \supset k$ , let  $Z_{\mathcal{A}, K}(\mathbf{t}) \in \mathbf{Q}(\mathbf{t})$  be the rational function with*

$$Z_{\mathcal{A}, K}(q_K^{-s}) = (1 - q_K^{-1})^d \cdot \zeta_{\mathcal{A} \otimes_{\mathfrak{o}} \mathfrak{O}_K}(s).$$

Then  $Z_{\mathcal{A}} := (Z_{\mathcal{A}, K})$  satisfies the assumptions in Theorem & Definition 6.2.

The **topological subalgebra zeta function** of  $\mathcal{A}$  is

$$\zeta_{\mathcal{A}, \text{top}}(\mathbf{s}) := Z_{\mathcal{A}, \text{top}}(\mathbf{s}) \in \mathbf{Q}(\mathbf{s}).$$

- (ii) *Let  $M$  be a free  $\mathfrak{o}$ -module of rank  $d$  and let  $\mathcal{E}$  be a subalgebra of  $\text{End}_{\mathfrak{o}}(M)$ . For a  $p$ -adic field  $K \supset k$ , let  $Z_{\mathcal{E} \curvearrowright M, K}(\mathbf{t}) \in \mathbf{Q}(\mathbf{t})$  be the rational function with*

$$Z_{\mathcal{E} \curvearrowright M, K}(q_K^{-s}) = (1 - q_K^{-1})^d \cdot \zeta_{(\mathcal{E} \otimes_{\mathfrak{o}} \mathfrak{O}_K) \curvearrowright (M \otimes_{\mathfrak{o}} \mathfrak{O}_K)}(s).$$

Then  $Z_{\mathcal{E} \curvearrowright M} := (Z_{\mathcal{E} \curvearrowright M, K})$  satisfies the assumptions in Theorem & Definition 6.2.

The **topological submodule zeta function** of  $\mathcal{E}$  acting on  $M$  is

$$\zeta_{\mathcal{E} \curvearrowright M, \text{top}}(\mathbf{s}) := Z_{\mathcal{E} \curvearrowright M, \text{top}}(\mathbf{s}) \in \mathbf{Q}(\mathbf{s}).$$

In view of Definition 2.1(ii), we define the **topological ideal zeta function** of  $\mathcal{A}$  to be  $\zeta_{\mathcal{A},\text{top}}^{\triangleleft}(\mathbf{s}) = \zeta_{\Omega(\mathcal{A}) \curvearrowright \mathcal{A},\text{top}}(\mathbf{s})$ .

**Remark 6.4.** Topological subalgebra zeta functions were first defined in greater generality by du Sautoy and Loeser [12]. Their definition of the topological subalgebra zeta function of a  $\mathbf{Z}$ -algebra  $\mathcal{A}$  of  $\mathbf{Z}$ -rank  $d$  coincides with  $d! \cdot Z_{\mathcal{A},\text{top}}(\mathbf{s} + d)$  in our notation; we note that the factor  $d!$  (a consequence of [12, Def. 7.2] and the remarks following it) seems to be missing from the examples in [12, §9].

The following simple observation will be useful for our computations, see §8.

**Lemma 6.5.** *Notation as in Theorem 6.3, the univariate rational functions  $\zeta_{\mathcal{A},\text{top}}(\mathbf{s}) \in \mathbf{Q}(\mathbf{s})$  and  $\zeta_{\mathcal{E} \curvearrowright M,\text{top}}(\mathbf{s}) \in \mathbf{Q}(\mathbf{s})$  both have degree  $\leq 0$  in  $\mathbf{s}$ .*

*Proof.* This follows from the explicit formula [12, Prop. 8.4] for topological zeta functions associated with “cone integrals”—or equivalently, the topological counterpart of [11, Cor. 3.2]. Indeed, these formulae express the topological zeta functions under consideration as  $\mathbf{Z}$ -linear combinations of rational functions of degree  $\leq 0$  in  $\mathbf{s}$ .  $\blacklozenge$

For all examples of topological subalgebra zeta functions known to the author, the degree of  $\zeta_{\mathcal{A},\text{top}}(\mathbf{s})$  is precisely  $-d$ , where  $d$  is the  $\mathfrak{o}$ -rank of  $\mathcal{A}$ , see [22, §8, Conj. I].

**Theorem & Definition 6.6** ([22, §5.4]; cf. [12, §6]). *Let  $G$  be a finitely generated torsion-free nilpotent group. Let  $\mathfrak{L}(G)$  be the associated Lie  $\mathbf{Q}$ -algebra under the Mal’cev correspondence. Choose an arbitrary  $\mathbf{Z}$ -subalgebra  $\mathcal{L} \subset \mathfrak{L}(G)$  which is finitely generated as a  $\mathbf{Z}$ -module and which spans  $\mathfrak{L}(G)$  over  $\mathbf{Q}$ . The **topological subgroup zeta function** and **topological normal subgroup zeta function** of  $G$  are  $\zeta_{G,\text{top}}(\mathbf{s}) := \zeta_{\mathcal{L},\text{top}}(\mathbf{s})$  and  $\zeta_{G,\text{top}}^{\triangleleft}(\mathbf{s}) := \zeta_{\mathcal{L},\text{top}}^{\triangleleft}(\mathbf{s})$ , respectively. These definitions do not depend on the choice of  $\mathcal{L}$ .*

## 6.2 Topological zeta functions associated with toric data

Let  $\mathcal{T}$  be a toric datum in  $n$  variables over  $k$  and let  $\beta$  be an  $m \times n$  matrix with entries in  $\mathbf{N}_0$ . The following is a special case of general results in  $p$ -adic integration following Denef’s fundamental paper [7], cf. [22, Rem. 4.7] and [22, Ex. 5.11(vi)]. Thus, for each  $p$ -adic field  $K \supset k$ , the zeta function  $Z_K^{\mathcal{T},\beta}(s_1, \dots, s_m)$  is rational in  $q_K^{-s_1}, \dots, q_K^{-s_m}$ . We may therefore regard each  $Z_K^{\mathcal{T},\beta}(s_1, \dots, s_m)$  as an element of  $\mathbf{Q}(t_1, \dots, t_m)$  via  $t_j \mapsto q_K^{-s_j}$ . After this identification, the collection of rational functions  $Z^{\mathcal{T},\beta} := (Z_K^{\mathcal{T},\beta})$  satisfies the assumptions in Theorem & Definition 6.2. Consequently, we obtain a topological zeta function  $Z_{\text{top}}^{\mathcal{T},\beta}(s_1, \dots, s_m) \in \mathbf{Q}(s_1, \dots, s_m)$  associated with  $\mathcal{T}$  and  $\beta$ .

**Lemma 6.7.** *Let  $\mathcal{T}$  and  $\beta$  be as above. Let  $\mathfrak{C}$  be a partition of  $\mathcal{T}$  (see Definition 5.4). Then  $Z_{\text{top}}^{\mathcal{T},\beta}(s_1, \dots, s_m) = \sum_{\mathcal{T}' \in \mathfrak{C}} Z_{\text{top}}^{\mathcal{T}',\beta}(s_1, \dots, s_m)$ .*  $\blacklozenge$

Hence, if we assume that the invariant ( $\clubsuit$ ) on p. 9 is preserved by both SIMPLIFY and REDUCE and that EVALUATETOPOLOGICALLY( $\mathcal{T}, \beta$ ) indeed computes  $Z_{\text{top}}^{\mathcal{T},\beta}(s_1, \dots, s_m)$ , then the correctness of Algorithm 4.1 follows from Lemma 6.7.



### 6.3 Torus factors

The group  $\mathrm{GL}_n(\mathbf{Z})$  admits a natural right-action on  $k[\mathbf{X}^{\pm 1}]$  by  $k$ -algebra automorphisms via  $(\mathbf{X}^\alpha)^A := \mathbf{X}^{\alpha A}$  for  $\alpha \in \mathbf{Z}^n$  and  $A \in \mathrm{GL}_n(\mathbf{Z})$ .

**Lemma 6.8** ([22, Lem. 6.1(i)]). *Let  $f_1, \dots, f_r \in k[\mathbf{X}^{\pm 1}]$  be non-zero Laurent polynomials. Let  $\mathcal{N} = \mathrm{New}(f_1 \cdots f_r)$  and  $d = \dim(\mathcal{N})$ . For  $1 \leq i \leq r$ , choose  $\alpha_i \in \mathrm{supp}(f_i)$ . Then there exists  $A \in \mathrm{GL}_n(\mathbf{Z})$  such that  $(\mathbf{X}^{-\alpha_i} f_i)^A \in k[X_1^{\pm 1}, \dots, X_d^{\pm 1}]$  for  $1 \leq i \leq r$ .*

Recall that Theorem 5.8 featured certain explicitly defined subvarieties of algebraic tori over  $k$ . The relevance of Lemma 6.8 is due to the following geometric consequence.

**Corollary 6.9.** *Write  $g_i := (\mathbf{X}^{-\alpha_i} f_i)^A$  for  $1 \leq i \leq r$ . Let  $V$  be the subvariety of  $\mathbf{T}_k^n$  defined by  $f_1 = \dots = f_r = 0$  and let  $U$  be the subvariety of  $\mathbf{T}_k^d$  defined by  $g_1 = \dots = g_r = 0$ . Then  $V \approx_k U \times_{\mathrm{Spec}(k)} \mathbf{T}_k^{n-d}$ .  $\blacklozenge$*

The proof of Lemma 6.8 given in [22] easily translates into an algorithm. Indeed, let  $M$  be the  $\mathbf{Z}$ -submodule of  $\mathbf{Z}^n$  generated by  $\bigcup_{i=1}^r \mathrm{supp}(\mathbf{X}^{-\alpha_i} f_i)$ . Then  $M$  has rank  $d$ , see the proof of [22, Lem. 6.1(i)]. Let  $B$  be any matrix over  $\mathbf{Z}$  (of size  $e \times n$ , say) whose rows span  $M$  over  $\mathbf{Z}$ . We may find  $C \in \mathrm{GL}_e(\mathbf{Z})$  and  $A \in \mathrm{GL}_n(\mathbf{Z})$  such that  $CBA$  is in Smith normal form. Evidently,  $A$  then satisfies the desired conditions in Lemma 6.8.

### 6.4 Rewriting Theorem 5.8

In [22], we used Lemma 6.8 to rewrite the explicit formulae in [22, Thm 4.10] in a shape compatible with Theorem & Definition 6.2. In order to be able to explicitly compute associated topological zeta functions, we now consider an algorithmic version of this rewriting process applied to the formula in Theorem 5.8.

Let  $\mathcal{T} = (\mathcal{C}_0; f_1, \dots, f_r)$  be a non-trivial regular toric datum. Let  $\beta$ ,  $V_J^\circ$ ,  $\mathcal{C}_0^J$ , and  $A_J(\beta)$  be as in Theorem 5.8. Let  $\mathcal{N} := \mathrm{New}(f_1 \cdots f_r)$ . Since  $\mathcal{T}$  is balanced, by Lemma 5.3, there exists a (unique) face  $\tau \subset \mathcal{N}$  such that  $\mathcal{C}_0 \subset N_\tau(\mathcal{N})$ ; in particular,  $\dim(\mathcal{C}_0) \leq n - \dim(\tau)$ . Let  $\tau = \tau_1 + \dots + \tau_r$  be the decomposition of  $\tau$  into faces  $\tau_i \subset \mathrm{New}(f_i)$ ; hence,  $\tau_i = \mathrm{New}(\mathrm{in}_{\mathcal{C}_0}(f_i))$  (cf. [22, Lem. 6.1(iii)]).

For  $J \subset \{1, \dots, r\}$ , let  $V_J$  be the subvariety of  $\mathbf{T}_k^n$  defined by  $\mathrm{in}_{\mathcal{C}_0}(f_j) = 0$  for all  $j \in J$ . Note that using the inclusion-exclusion principle, we may replace  $\#\tilde{V}_J^\circ(\mathfrak{D}_K/\mathfrak{P}_K)$  in Theorem 5.8 by  $\sum_{J \subset T \subset \{1, \dots, r\}} (-1)^{|T|+|J|} \cdot \#\tilde{V}_T(\mathfrak{D}_K/\mathfrak{P}_K)$ .

Again, let  $J \subset \{1, \dots, r\}$ . Write  $d(J) := \dim(\sum_{j \in J} \tau_j)$ . Equivalently,  $d(J)$  is the dimension of  $\sum_{j \in J} \mathrm{New}(\mathrm{in}_{\mathcal{C}_0}(f_j)) = \mathrm{New}(\prod_{j \in J} \mathrm{in}_{\mathcal{C}_0}(f_j))$ . By Lemma 6.8, we may thus construct  $B_J \in \mathrm{GL}_n(\mathbf{Z})$  and non-zero  $g_j \in k[X_1^{\pm 1}, \dots, X_{d(J)}^{\pm 1}]$  such that  $g_j^{-1} \mathrm{in}_{\mathcal{C}_0}(f_j)^{B_J}$  is a Laurent monomial for each  $j \in J$ . Let  $U_J$  be the subvariety of  $\mathbf{T}_k^{d(J)}$  defined by  $g_j = 0$  for all  $j \in J$  so that  $V_J \approx_k U_J \times_{\mathrm{Spec}(k)} \mathbf{T}_k^{n-d(J)}$  (see Corollary 6.9). Finally, define

$$W_J(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) := \mathbf{q}^{-n}(\mathbf{q} - 1)^{n - \dim(\tau) + |J|} \cdot |\mathcal{C}_0^J|^{A_J(\beta)}(\mathbf{q}^{-1}, \mathbf{t}_1, \dots, \mathbf{t}_m) \in \mathbf{Q}(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m). \quad (6.2)$$

**Proposition 6.10.** *Notation as above; in particular, let  $\mathcal{T} = (\mathcal{C}_0; f_1, \dots, f_r)$  be regular. For all  $p$ -adic fields  $K \supset k$ , unless  $\mathfrak{p} = \mathfrak{o} \cap \mathfrak{P}_K$  belongs to some finite exceptional set,*

$$Z_K^{\mathcal{T}, \beta}(s_1, \dots, s_m) = \sum_J \left( \sum_{J \subset T} (-1)^{|J|+|T|} \cdot \# \bar{U}_T(\mathfrak{D}_K / \mathfrak{P}_K) \cdot (q_K - 1)^{\dim(\tau) - d(T)} \right) \cdot W_J(q_K, q_K^{-s_1}, \dots, q_K^{-s_m}), \quad (6.3)$$

where  $J$  and  $T$  range over subsets of  $\{1, \dots, r\}$  and  $s_1, \dots, s_m \in \mathbf{C}$  with  $\operatorname{Re}(s_j) \geq 0$ .  $\blacklozenge$

The two crucial features of Proposition 6.10 compared with Theorem 5.8 are

- (i) the  $U_T$  are embedded as closed (instead of locally closed) subvarieties of tori and
- (ii) each  $W_J(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  belongs to the algebra  $\mathbf{M}$  from §6.1 (Corollary 6.13).

## 6.5 Computing formal reductions modulo $q - 1$

We show that each  $W_J$  in Proposition 6.10 belongs to the  $\mathbf{Q}$ -algebra  $\mathbf{M}$  from §6.1. While this statement alone is merely a special case of [22, Lem. 6.9(i)], the proof given here provides an algorithm for computing  $[W](s_1, \dots, s_m)$  (see Notation 6.1(ii)).

**Lemma 6.11.** *Let  $\mathcal{B}_0 \subset \mathbf{R}_{\geq 0}^r$  be a non-empty rational half-open cone of dimension  $d$ . Let  $A$  be an  $r \times (m+1)$ -matrix with entries in  $\mathbf{N}_0$  and suppose that the first column of  $A$  is  $(1, \dots, 1)^\top$ . Then  $W(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) := (\mathbf{q} - 1)^d |\mathcal{B}_0|^A(\mathbf{q}^{-1}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  belongs to the algebra  $\mathbf{M}$  from §6.1. Moreover,  $-d \leq \deg_{s_j}([W](s_1, \dots, s_m)) \leq 0$  for  $j = 1, \dots, m$ .*

*Proof.* Let  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$  consist of independent variables over  $\mathbf{Q}$ . We regard generating functions of rational half-open cones in  $\mathbf{R}_{\geq 0}^r$  as elements of  $\mathbf{Q}(\boldsymbol{\lambda})$ , cf. §5.4.

Let  $\mathcal{F}$  be a triangulation of the closure  $\bar{\mathcal{B}}_0$  of  $\mathcal{B}_0$  into simplicial cones. Let  $\sigma \in \mathcal{F}$ , say  $\sigma = \operatorname{cone}(\varrho_1, \dots, \varrho_e)$ , where  $\varrho_1, \dots, \varrho_e \in \mathbf{N}_0^r$  are primitive vectors and  $e = \dim(\sigma)$ . Let  $\Pi(\sigma) = \{a_1 \varrho_1 + \dots + a_e \varrho_e : 0 \leq a_i < 1\}$ . Then  $|\sigma| = (\sum_{\beta \in \Pi(\sigma) \cap \mathbf{Z}^r} \boldsymbol{\lambda}^\beta) / \prod_{i=1}^e (1 - \boldsymbol{\lambda}^{\varrho_i})$ . Write  $\mathbf{1} = (1, \dots, 1)$  and  $A = [\mathbf{1}^\top, \alpha_1^\top, \dots, \alpha_m^\top]$ . Therefore

$$|\sigma|^A(\mathbf{q}^{-1}, \mathbf{t}_1, \dots, \mathbf{t}_m) = \frac{\sum_{\beta \in \Pi(\sigma) \cap \mathbf{Z}^r} \mathbf{q}^{-\langle \mathbf{1}, \beta \rangle} \mathbf{t}_1^{\langle \alpha_1, \beta \rangle} \dots \mathbf{t}_m^{\langle \alpha_m, \beta \rangle}}{\prod_{i=1}^e \left( 1 - \mathbf{q}^{-\langle \mathbf{1}, \varrho_i \rangle} \mathbf{t}_1^{\langle \alpha_1, \varrho_i \rangle} \dots \mathbf{t}_m^{\langle \alpha_m, \varrho_i \rangle} \right)}.$$

For  $b \in \mathbf{N}$  and  $\mathbf{a} = (a_1, \dots, a_m) \in \mathbf{N}_0^m$ , let  $W_{\mathbf{a}, b}(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) := \frac{\mathbf{q}^{-1}}{1 - \mathbf{q}^{-b} \mathbf{t}_1^{a_1} \dots \mathbf{t}_m^{a_m}}$ . Then  $W_{\mathbf{a}, b}(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) \in \mathbf{M}$  and  $[W_{\mathbf{a}, b}](s_1, \dots, s_m) = 1/(a_1 s_1 + \dots + a_m s_m + b)$ . We conclude that  $Z_\sigma(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) := (\mathbf{q} - 1)^d \cdot |\sigma|^A(\mathbf{q}^{-1}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  belongs to  $\mathbf{M}$ . Moreover, if  $\dim(\sigma) = e < d = \dim(\mathcal{B}_0)$ , then  $[Z_\sigma](s_1, \dots, s_m) = 0$ . If, on the other hand,  $d = e$ , then

$$[Z_\sigma](s_1, \dots, s_m) = \frac{\#\Pi(\sigma)}{\prod_{i=1}^e \langle \varrho_i A, (1, s_1, \dots, s_m) \rangle}; \quad (6.4)$$

we note that  $\#\Pi(\sigma)$  is the usual multiplicity of the simplicial cone  $\sigma$ , see [6, Prop. 11.1.8].

As a consequence of the inclusion-exclusion principle, we may write  $W(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) = \sum_{\sigma \in \mathcal{F}} c_\sigma Z_\sigma(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$ , where  $c_\sigma \in \mathbf{Z}$  and  $c_\sigma = 1$  whenever  $\dim(\sigma) = d$ . In particular,

$W(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m) \in \mathbf{M}$  and  $[W](\mathbf{s}_1, \dots, \mathbf{s}_m) = \sum_{\sigma} [Z_{\sigma}](\mathbf{s}_1, \dots, \mathbf{s}_m)$ , where the sum is taken over those  $\sigma \in \mathcal{F}$  with  $\dim(\sigma) = d$  only. In the expression for  $[Z_{\sigma}](\mathbf{s}_1, \dots, \mathbf{s}_m)$  given in (6.4), the numerator is a positive constant and the denominator is a product of  $\dim(\sigma)$  factors of the form  $a_1 \mathbf{s}_1 + \dots + a_m \mathbf{s}_m + b$  for  $a_1, \dots, a_m \in \mathbf{N}_0$  and  $b \in \mathbf{N}$ . By the non-negativity of all these numbers, the degree of  $[W](\mathbf{s}_1, \dots, \mathbf{s}_m)$  in  $\mathbf{s}_j$  is simply the maximal degree of any  $[Z_{\sigma}](\mathbf{s}_1, \dots, \mathbf{s}_m)$  in  $\mathbf{s}_j$  for  $\sigma \in \mathcal{F}$  with  $\dim(\sigma) = d$ .  $\blacklozenge$

**Remark 6.12.**

- (i) In [9, §5], Denef and Loeser gave an explicit convex-geometric formula for the topological zeta function associated with a suitably non-degenerate polynomial. In view of the  $p$ -adic formulae of Denef and Hoornaert [8], the explicit descriptions of the rational functions  $J(\tau, s) \in \mathbf{Q}(s)$  in terms of triangulations in [9] can be regarded as a special case of Lemma 6.11.
- (ii) The proof of Lemma 6.11 shows that the rational function  $[W](\mathbf{s}_1, \dots, \mathbf{s}_m)$  only depends on the closure of  $\mathcal{B}_0$ .

**Corollary 6.13.** *Notation as in §6.4. For each  $J \subset \{1, \dots, r\}$ , the rational function  $W_J(\mathbf{q}, \mathbf{t}_1, \dots, \mathbf{t}_m)$  belongs to  $\mathbf{M}$ . Moreover,  $[W_J](\mathbf{s}_1, \dots, \mathbf{s}_m) = 0$  if and only if  $\dim(\mathcal{C}_0^J) < n - \dim(\tau) + |J|$ .*

*Proof.*  $\dim(\mathcal{C}_0) \leq n - \dim(\tau)$  and  $\mathcal{C}_0^J \subset \mathcal{C}_0 \times \mathbf{R}_{>0}^J$  whence  $\dim(\mathcal{C}_0^J) \leq n - \dim(\tau) + |J|$ .  $\blacklozenge$

## 6.6 Computing Euler characteristics

Let  $U$  be the closed subvariety of  $\mathbf{T}_k^n$  defined by  $f_1 = \dots = f_r = 0$  for  $f_1, \dots, f_r \in k[\mathbf{X}^{\pm 1}]$ . The typical example to bear in mind is the case where  $f_1, \dots, f_r$  are the initial forms of a non-empty regular toric datum. We now consider the computation of the topological Euler characteristic  $\chi(U(\mathbf{C}))$ . The function EVALUATETOPOLOGICALLY (see §6.7) will rely on our ability to compute these numbers.

**General methods.** Aluffi [2] described an algorithm for computing the topological Euler characteristic of a not necessarily smooth projective variety in characteristic zero based on the computation of so-called Chern-Schwartz-MacPherson classes; for recent developments, see [16, 19]. In principle, such general algorithms can be used to compute  $\chi(U(\mathbf{C}))$  from above. Indeed, after clearing denominators, we may assume that  $f_1, \dots, f_r \in k[\mathbf{X}]$ . Let  $\tilde{f}_i \in k[X_0, \dots, X_{n+1}]$  denote the homogenisation of  $f_i$ . Let the subvarieties  $V, W \subset \mathbf{P}_k^n$  be defined by  $\tilde{f}_1 = \dots = \tilde{f}_r = 0$  and  $\tilde{f}_1 = \dots = \tilde{f}_r = X_0 \cdots X_n = 0$ , respectively. Then  $U \approx_k V \setminus W$  and so  $\chi(U(\mathbf{C})) = \chi(V(\mathbf{C})) - \chi(W(\mathbf{C}))$ ; cf. [2, §2.8].

In practice, while implementations of [2, 16, 19] exist, these methods are usually too costly for our applications to the computation of topological zeta functions. For example, the computation of  $\zeta_{\text{Fil}_4, \text{top}}(\mathbf{s})$  previously announced in [22, §7.3] involves the Euler characteristics of thousands of subvarieties of  $\mathbf{T}_{\mathbf{Q}}^{15}$ . In our implementation (see §8), we therefore attempt to compute Euler characteristics using special-purpose methods.

**The Bernstein-Khovanskii-Kushnirenko Theorem.** As we already exploited in [22, §6], if  $(f_1, \dots, f_r)$  is non-degenerate in the sense of Khovanskii [20, §2], then [20, §3, Thm 2] provides an explicit formula for  $\chi(U(\mathbf{C}))$  in terms of various mixed volumes associated with the Newton polytopes of  $f_1, \dots, f_r$ . Since  $U$ , and hence  $\chi(U(\mathbf{C}))$ , only depends on the radical of the ideal generated by  $f_1, \dots, f_r$  within  $k[\mathbf{X}^{\pm 1}]$ , when  $(f_1, \dots, f_r)$  is degenerate in Khovanskii's sense, we can try to use standard techniques such as multivariate polynomial division (after clearing denominators) and saturation to “simplify”  $(f_1, \dots, f_r)$ , e.g. by reducing  $\sum_{i=1}^r |\text{supp}(f_i)|$ .

**Decomposing subvarieties of tori.** If  $(f_1, \dots, f_r)$  remains degenerate after applying the simplification steps indicated above, we try to decompose  $U$  as follows. Suppose that after renumbering of  $1, \dots, r$  (or, more generally, a suitable application of a matrix from  $\text{GL}_n(\mathbf{Z})$  as in §6.3) and rescaling of  $f_1$  by Laurent monomials (which does not change  $U$ ), we have  $f_1 = X_n - w$  for  $w \in k[X_1^{\pm 1}, \dots, X_{n-1}^{\pm 1}]$ . Let  $V \subset \mathbf{T}_k^{n-1}$  be the subvariety defined by  $f_2(X_1, \dots, X_{n-1}, w) = \dots = f_r(X_1, \dots, X_{n-1}, w) = 0$  and let  $W \subset V$  be defined by  $w = 0$ . Then  $U \approx_k V \setminus W$  and we can recursively try to compute  $\chi(U(\mathbf{C})) = \chi(V(\mathbf{C})) - \chi(W(\mathbf{C}))$  using the techniques mentioned above.

In practice, combining these methods often suffices to compute Euler characteristics in Algorithm 4.1.

## 6.7 An algorithm for computing topological zeta functions associated with regular toric data

The following is a topological version of Proposition 6.10.

**Proposition 6.14.** *Notation as in Proposition 6.10; in particular,  $\mathcal{T} = (\mathcal{C}_0; f_1, \dots, f_r)$  is a regular toric datum. Then*

$$Z_{\text{top}}^{\mathcal{T}, \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m) = \sum_{\substack{J \subset T \subset \{1, \dots, r\}, \\ n - d(T) + |J| = \dim(\mathcal{C}_0^J)}} (-1)^{|J| + |T|} \cdot \chi(U_T(\mathbf{C})) \cdot [W_J](\mathbf{s}_1, \dots, \mathbf{s}_m).$$

*Proof.* Let  $J \subset T$ . Proposition 6.10 and Corollary 6.13 show that  $Z_{\text{top}}^{\mathcal{T}, \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m)$  is the sum of  $(-1)^{|J| + |T|} \cdot \chi(U_T(\mathbf{C})) \cdot [W_J](\mathbf{s}_1, \dots, \mathbf{s}_m)$  over pairs  $J \subset T$  with  $d(T) = \dim(\tau)$  and  $n - \dim(\tau) + |J| = \dim(\mathcal{C}_0^J)$ . The latter two conditions are both satisfied if and only if  $n - d(T) + |J| = \dim(\mathcal{C}_0^J)$  since  $d(T) \leq \dim(\tau)$  and  $n - \dim(\tau) + |J| \geq \dim(\mathcal{C}_0^J)$ . ♦

The Euler characteristics  $\chi(U_T(\mathbf{C}))$  can be determined as in §6.6, while the rational functions  $[W_J](\mathbf{s}_1, \dots, \mathbf{s}_m)$  may be computed as explained in §6.5. We obtain the following algorithm.

**The function EVALUATETOPologically.** We let EVALUATETOPologically denote the function which, given a regular toric datum  $\mathcal{T}$  in  $n$  variables over  $k$  and a matrix  $\beta \in \text{M}_{m \times n}(\mathbf{N}_0)$ , computes  $Z_{\text{top}}^{\mathcal{T}, \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m)$  using Proposition 6.14. Specifically, for each

$J \subset \{1, \dots, r\}$ , we use §6.3 and §6.6 to first compute  $e_J := \sum (-1)^{|J|+|T|} \cdot \chi(U_T(\mathbf{C}))$ , the sum being over those  $T \supset J$  with  $n - d(T) + |J| = \dim(\mathcal{C}_0^J)$  as in Proposition 6.14. Only if  $e_J$  turns out to be non-zero, do we proceed to compute  $\lfloor W_J \rfloor(\mathbf{s}_1, \dots, \mathbf{s}_m)$  using a triangulation of the closure of  $\mathcal{C}_0^J$  as in the proof of Lemma 6.11.

**Remark 6.15.**

- (i) The topological zeta functions that we seek to compute can be written as univariate specialisations of topological zeta functions associated with toric data, see Remark 3.4. In practice, we avoid the costly multivariate rational function arithmetic altogether and apply these specialisations directly in the triangulation step of EVALUATETOPOLOGICALLY; for a theoretical justification, use [22, Rem. 5.15].
- (ii) If we ignore the simplification step in line 4 of Algorithm 4.1, then, at this point, we have obtained an algorithmic version of [22, Thm 6.7] (restricted to the integrals considered here). Namely, let  $(\mathcal{C}_0; \mathbf{f})$  be a toric datum as in Algorithm 4.1 and suppose that  $\mathbf{f}$  is globally non-degenerate in the sense of [22, Def. 4.2(ii)]. It follows from [22, Lem. 6.1] that each  $\mathcal{T} \in \text{BALANCE}(\mathcal{C}_0; \mathbf{f})$  is regular and that the defining polynomials of the varieties  $U_T$  in EVALUATETOPOLOGICALLY( $\mathcal{T}, \beta$ ) satisfy Khovanskii's non-degeneracy conditions; the computation of  $e_J$  as part of EVALUATETOPOLOGICALLY is then a direct implementation of [22, Prop. 6.5].

## 7 Simplification and reduction

We now describe the remaining two functions SIMPLIFY and REDUCE in Algorithm 4.1.

### 7.1 Weak and strong equivalence of toric data

**Definition 7.1.** Let  $(\mathcal{C}_0; \mathbf{f})$  and  $(\mathcal{D}_0; \mathbf{g})$  be toric data over  $k$ .

- (i) We say that  $(\mathcal{C}_0; \mathbf{f})$  and  $(\mathcal{D}_0; \mathbf{g})$  are **strongly equivalent** if  $\mathcal{C}_0 = \mathcal{D}_0$  and there exists a finite  $S \subset \text{Spec}(\mathfrak{o})$  such that if  $K \supset k$  is a  $p$ -adic field with  $\mathfrak{o} \cap \mathfrak{P}_K \notin S$ , then  $\|\mathbf{f}(\mathbf{x})\|_K = \|\mathbf{g}(\mathbf{x})\|_K$  for all  $\mathbf{x} \in \mathbf{T}^n(K)$  with  $\nu_K(\mathbf{x}) \in \mathcal{C}_0$ .
- (ii) We say that  $(\mathcal{C}_0; \mathbf{f})$  and  $(\mathcal{D}_0; \mathbf{g})$  are **weakly equivalent** if there is a finite  $S \subset \text{Spec}(\mathfrak{o})$  such that  $(\mathcal{C}_0; \mathbf{f})_K = (\mathcal{D}_0; \mathbf{g})_K$  for all  $p$ -adic fields  $K \supset k$  with  $\mathfrak{o} \cap \mathfrak{P}_K \notin S$ .

Strong equivalence implies weak one but the converse is false; for example,  $(\mathbf{R}_{\geq 0}; X_1^{-1})$  and  $(\mathbf{R}_{\geq 0}; X_1^{-1} - X_1)$  are weakly equivalent but not strongly so. Theorem & Definition 6.2 yields the following.

**Lemma 7.2.** *Let  $\mathcal{T}$  and  $\mathcal{T}'$  be weakly equivalent toric data in  $n$  variables over  $K$  and let  $\beta \in M_{m \times n}(\mathbf{N}_0)$ . Then  $Z_{\text{top}}^{\mathcal{T}, \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m) = Z_{\text{top}}^{\mathcal{T}', \beta}(\mathbf{s}_1, \dots, \mathbf{s}_m)$ .  $\blacklozenge$*

We now collect some instances of these equivalences in a form that resembles Gaussian elimination and the multivariate polynomial division algorithm (see e.g. [1, §1.5]). By a **term**, we mean a Laurent polynomial of the form  $c\mathbf{X}^\alpha$ , where  $c \in k^\times$  and  $\alpha \in \mathbf{Z}^n$ . Given

a rational half-open cone  $\mathcal{C}_0 \subset \mathbf{R}_{\geq 0}^n$ , its dual  $\mathcal{C}_0^* = \{\omega \in \mathbf{R}^n : \langle \alpha, \omega \rangle \geq 0 \text{ for all } \alpha \in \mathcal{C}_0\}$  is a rational *closed* cone which contains  $\mathbf{R}_{\geq 0}^n$ . The unit group  $k[\mathcal{C}_0^* \cap \mathbf{Z}^n]^\times$  of  $k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$  (see §5.4) consists precisely of those terms  $c\mathbf{X}^\alpha$  with  $c \in k^\times$  and  $\alpha \in \mathcal{C}_0^\perp \cap \mathbf{Z}^n$ . In the following, we assume that  $r \in \mathbf{N}_0$  is large enough for the statements given to make sense.

**Lemma 7.3.** *Let  $(\mathcal{C}_0; \mathbf{f}) = (\mathcal{C}_0; f_1, \dots, f_r)$  be a toric datum over  $k$ . We let  $\sim_s$  and  $\sim_w$  signify strong and weak equivalence, respectively. Then:*

- (S0)  $(\mathcal{C}_0; f_1, \dots, f_r, 0) \sim_s (\mathcal{C}_0; f_1, \dots, f_r)$ .
- (S1)  $(\mathcal{C}_0; f_1, \dots, f_r) \sim_s (\mathcal{C}_0; f_{1\sigma}, \dots, f_{r\sigma})$  for any permutation  $\sigma \in \text{Sym}(r)$ .
- (S2) If  $u \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]^\times$  and  $v \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ , then  $(\mathcal{C}_0; \mathbf{f}) \sim_s (\mathcal{C}_0; uf_1 + vf_2, f_2, \dots, f_r)$ .
- (S3) If  $f_1$  is  $\mathcal{C}_0$ -balanced and  $\text{in}_{\mathcal{C}_0}(f_1)$  is a term, then  $(\mathcal{C}_0; \mathbf{f}) \sim_s (\mathcal{C}_0; \text{in}_{\mathcal{C}_0}(f_1), f_2, \dots, f_r)$ .
- (W1) If  $v \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ , then  $(\mathcal{C}_0; \mathbf{f}) \sim_w (\mathcal{C}_0; f_1 + v, f_2, \dots, f_r)$ .
- (W2) If  $f_1 = c\mathbf{X}^\alpha$  for  $c \in k^\times$  and  $\alpha \in \mathbf{Z}^n$ , then  $(\mathcal{C}_0; \mathbf{f}) \sim_w (\mathcal{C}_0 \cap \{\alpha\}^*; f_2, \dots, f_r)$ .

*Proof.* (S0) and (S1) are obvious. Let  $K \supset k$  be a  $p$ -adic field. Let  $\omega \in \mathbf{Z}^n$  and  $\mathbf{x} \in \mathbf{T}^n(K)$  with  $\nu_K(\mathbf{x}) = \omega$ . Write  $\mathbf{x} = (\pi_K^{\omega_1} u_1, \dots, \pi_K^{\omega_n} u_n)$  for  $\mathbf{u} \in \mathbf{T}^n(\mathfrak{O}_K)$ . Then, for any non-zero  $g \in \mathfrak{O}_K[\mathbf{X}^{\pm 1}]$ , we have  $g(\mathbf{x}) = \pi_K^{\langle \alpha, \omega \rangle} \cdot (\text{in}_\omega(g)(\mathbf{u}) + \mathcal{O}(\pi_K))$ , where  $\alpha \in \text{supp}(\text{in}_\omega(g))$  is arbitrary. Hence, if  $u \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]^\times$ ,  $v \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ , and  $\mathbf{x} \in \mathbf{T}^n(K)$  with  $\nu_K(\mathbf{x}) \in \mathcal{C}_0$ , then  $|u(\mathbf{x})|_K = 1$  and  $|v(\mathbf{x})|_K \leq 1$ , provided that the unique non-zero coefficient of  $u$  is a  $\mathfrak{P}_K$ -adic unit and all coefficients of  $v$  are  $\mathfrak{P}_K$ -adic integers. (W1) is now obvious. (S2) follows since if  $a, b \in K$  and  $e \in \mathfrak{O}_K$ , then  $\|a, b\|_K = \|a + eb, b\|_K$ . Indeed, if  $|a|_K \leq |eb|_K$ , then  $|a|_K, |a + eb|_K \leq |b|_K$ ; if, on the other hand,  $|a|_K > |eb|_K$ , then  $|a + eb|_K = |a|_K$ . For (S3) and (W2), let  $f_1$  be  $\mathcal{C}_0$ -balanced with  $\text{in}_{\mathcal{C}_0}(f_1) = c\mathbf{X}^\alpha$ , where  $c \in k^\times$  and  $\alpha \in \mathbf{Z}^n$ ; we may assume that  $c \in \mathfrak{O}_K^\times$  and  $f_1 \in \mathfrak{O}_K[\mathbf{X}^{\pm 1}]$ . Then for  $x \in \mathbf{T}^n(K)$  with  $\nu_K(\mathbf{x}) = \omega \in \mathcal{C}_0 \cap \mathbf{Z}^n$ , we have  $|f_1(\mathbf{x})|_K = q_K^{-\langle \alpha, \omega \rangle} = |c\mathbf{x}^\alpha|_K$ . Hence,  $|f_1(\mathbf{x})|_K \leq 1$  if and only if  $\langle \alpha, \omega \rangle \geq 0$ .  $\blacklozenge$

## 7.2 Simplification

What we call simplification is the systematic application of Lemma 7.3 (with the exception of operation (S2), see Proposition 7.5) to toric data.

**Definition 7.4.** A toric datum  $(\mathcal{C}_0; f_1, \dots, f_r)$  is **simple** if the following conditions are satisfied for  $i = 1, \dots, r$ :

- (i)  $f_i \neq 0$  and no term of  $f_i$  lies in  $k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ .
- (ii) If  $f_i f_j^{-1} \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$  for  $1 \leq j \leq r$ , then  $i = j$ .
- (iii) If  $f_i$  is  $\mathcal{C}_0$ -balanced, then  $\text{in}_{\mathcal{C}_0}(f_i)$  consists of at least two terms.

**The function SIMPLIFY.** We now describe the function SIMPLIFY in Algorithm 4.1. Given a toric datum  $(\mathcal{C}_0; f_1, \dots, f_r)$ , we remove those terms of each  $f_i$  that lie in  $k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ . We then discard those  $f_j$  with  $f_j = 0$  altogether. Next, if  $f_i f_j^{-1} \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$  for  $i \neq j$ , then we discard  $f_i$ . Finally, if some  $f_i$  is  $\mathcal{C}_0$ -balanced with  $\text{in}_{\mathcal{C}_0}(f_i) = c\mathbf{X}^\alpha$  (where  $c \in k^\times$ ), then we discard  $f_i$  and replace  $\mathcal{C}_0$  by  $\mathcal{C}_0 \cap \{\alpha\}^*$ . Since shrinking  $\mathcal{C}_0$  enlarges its dual  $\mathcal{C}_0^*$ , further terms might now become redundant. We therefore repeatedly apply the above process until  $(\mathcal{C}_0; f_1, \dots, f_r)$  stabilises. As each non-trivial operation decreases  $r + \sum_{i=1}^r \#\text{supp}(f_i)$ , after finitely many steps, we obtain a simple toric datum which is weakly equivalent to the original  $(\mathcal{C}_0; f_1, \dots, f_r)$  by Lemma 7.3.

The reason we only made very limited use of Lemma 7.3(S2) is to ensure the following:

**Proposition 7.5.** *Let  $\mathcal{T}$  be a toric datum.*

- (i) *If  $\mathcal{T}$  is balanced, then so is  $\text{SIMPLIFY}(\mathcal{T})$ .*
- (ii) *If  $\mathcal{T}$  is regular, then so is  $\text{SIMPLIFY}(\mathcal{T})$ .*

*Proof.* Let  $\mathcal{T} = (\mathcal{C}_0; \mathbf{f})$  be non-trivial. The properties of being balanced or regular are preserved if we discard polynomials, shrink  $\mathcal{C}_0$ , or remove non-initial terms. Let  $f \in \mathbf{f}$  be  $\mathcal{C}_0$ -balanced and suppose that  $\text{in}_{\mathcal{C}_0}(f)$  contains a term  $c\mathbf{X}^\alpha$  with  $c \in k^\times$  and  $\alpha \in \mathcal{C}_0^*$ . Then  $0 \leq \langle \alpha, \omega \rangle \leq \langle \beta, \omega \rangle$  for all  $\beta \in \text{supp}(f)$  and  $\omega \in \mathcal{C}_0$  whence  $\text{SIMPLIFY}(\mathcal{T})$  will discard  $f$  entirely.  $\blacklozenge$

### 7.3 Reduction

We now describe the “reduction step” in Algorithm 4.1. The function REDUCE takes as input a balanced (Definition 5.1(ii)) and simple (Definition 7.4) toric datum  $\mathcal{T} = (\mathcal{C}_0; f_1, \dots, f_r)$  which is singular (Definition 5.5). We therefore cannot directly use EVALUATE TOPOLOGICALLY from §6.7 to compute the associated topological zeta function. Our goal is to construct and return a partition  $\mathfrak{C}$  of  $\mathcal{T}$ . Ideally, we would like  $\mathfrak{C}$  to consist of regular toric data but our immediate goal is more modest: we systematically construct *some* non-trivial partition  $\mathfrak{C}$  in the hope that repeated further applications of BALANCE, SIMPLIFY, and REDUCE to its members in the main loop of Algorithm 4.1 will eventually produce regular toric data only. Success of this procedure is not guaranteed and we need to allow REDUCE to fail (at which point Algorithm 4.1 will fail too) in order to guarantee termination.

**Reduction candidates.** We begin by isolating a source of the singularity of  $\mathcal{T}$ . Namely, the method for regularity testing in §5.3 readily provides us with an inclusion-minimal set  $J \subset \{1, \dots, r\}$  such that the Jacobian matrix of  $(\text{in}_{\mathcal{C}_0}(f_j))_{j \in J}$  has rank less than  $|J|$  at some point  $\mathbf{u} \in \mathbf{T}^n(\bar{k})$  with  $\text{in}_{\mathcal{C}_0}(f_j)(\mathbf{u}) = 0$  for all  $j \in J$ . After renumbering  $f_1, \dots, f_r$ , we may assume that  $J = \{1, \dots, e\}$ . If  $e = 1$ , then we give up and let REDUCE fail. Suppose that  $e \geq 2$ . By a **reduction candidate** for  $\mathcal{T}$  we mean a quadruple  $(i, j, t_i, t_j)$ , where  $1 \leq i < j \leq e$ ,  $t_i$  is a term of  $\text{in}_{\mathcal{C}_0}(f_i)$ , and  $t_j$  is a term of  $\text{in}_{\mathcal{C}_0}(f_j)$ .

**Performing reduction.** Let  $(i, j, t_i, t_j)$  be a reduction candidate for  $\mathcal{T}$ . Let  $\alpha_i$  and  $\alpha_j$  denote the exponent vectors of the monomial in  $t_i$  and  $t_j$ , respectively. We decompose  $\mathcal{C}_0$  into two pieces  $\mathcal{C}_0^{\leq} := \mathcal{C}_0 \cap \{\alpha_j - \alpha_i\}^*$  and  $\mathcal{C}_0^{>} := \mathcal{C}_0 \setminus \mathcal{C}_0^{\leq}$ , both of which are themselves rational half-open cones. Note that the restriction of the linear form  $\langle \alpha_i, - \rangle$  to  $\mathcal{C}_0$  only depends on  $f_i$  and not on the chosen term  $t_i$ , and similarly for  $\alpha_j$ . In particular, the decomposition  $\mathcal{C}_0 = \mathcal{C}_0^{\leq} \cup \mathcal{C}_0^{>}$  only depends on  $(i, j)$ . Define toric data

$$\begin{aligned}\mathcal{T}^{\leq} &:= \left( \mathcal{C}_0^{\leq}; f_1, \dots, f_{j-1}, f_j - \frac{t_j}{t_i} f_i, f_{j+1}, \dots, f_r \right), \\ \mathcal{T}^{>} &:= \left( \mathcal{C}_0^{>}; f_1, \dots, f_{i-1}, f_i - \frac{t_i}{t_j} f_j, f_{i+1}, \dots, f_r \right).\end{aligned}$$

By construction,  $t_j/t_i \in k[(\mathcal{C}_0^{\leq})^* \cap \mathbf{Z}^n]$  and  $t_i/t_j \in k[(\mathcal{C}_0^{>})^* \cap \mathbf{Z}^n]$  so Lemma 7.3(S2) shows that  $\{\mathcal{T}^{\leq}, \mathcal{T}^{>}\}$  is a partition of  $\mathcal{T}$ . Having chosen a reduction candidate  $(i, j, t_i, t_j)$ , we let  $\text{REDUCE}(\mathcal{T})$  return  $\{\mathcal{T}^{\leq}, \mathcal{T}^{>}\}$ .

The name “reduction” given to the procedure described here is due to the similarity to reduction steps in the theory of Gröbner bases, see e.g. [1, §1.5]. There are, however, substantial differences between the two procedures. Most importantly, the role of divisibility relations “ $t_i \mid t_j$ ” between terms in polynomial algebras in the classical setting is replaced by an integrality condition “ $t_j/t_i \in k[\mathcal{C}_0^* \cap \mathbf{Z}^n]$ ” for Laurent terms. In the present setting, we can enforce arbitrary divisibility relations of this form by cutting  $\mathcal{C}_0$  in half—at the cost of having to consider the opposite relation as well.

**Finding reduction candidates.** It remains to explain a strategy for choosing a reduction candidate  $(i, j, t_i, t_j)$  for  $\mathcal{T}$ . This is the most critical part of the entire reduction step and it may well fail. We use a greedy approach. Define the **weight** of a balanced toric datum  $\mathcal{T}' = (\mathcal{D}_0; g_1, \dots, g_u)$  to be  $w(\mathcal{T}') := \sum_{d=1}^u \#\text{supp}(\text{in}_{\mathcal{D}_0}(g_d))$ . For each reduction candidate  $(i, j, t_i, t_j)$ , we construct the associated partition  $\{\mathcal{T}^{\leq}, \mathcal{T}^{>}\}$  of  $\mathcal{T}$  as indicated above. Using  $\text{BALANCE}$  and  $\text{SIMPLIFY}$ , we then further refine this partition to produce a partition,  $\mathfrak{C}(i, j, t_i, t_j)$  say, of  $\mathcal{T}$  which consists of balanced and simple toric data. Let  $\mathfrak{C}'(i, j, t_i, t_j) \subset \mathfrak{C}(i, j, t_i, t_j)$  be the subset of singular toric data. If  $\mathfrak{C}'(i, j, t_i, t_j) = \emptyset$  for some  $(i, j, t_i, t_j)$ , then we use such a quadruple as our reduction candidate. Otherwise, we choose  $(i, j, t_i, t_j)$  such that  $(\sum_{\mathcal{T}' \in \mathfrak{C}'(i, j, t_i, t_j)} w(\mathcal{T}')) / |\mathfrak{C}'(i, j, t_i, t_j)|$  is minimal. In practice, we then of course let  $\text{REDUCE}(\mathcal{T})$  return  $\mathfrak{C}(i, j, t_i, t_j)$  instead of  $\{\mathcal{T}^{\leq}, \mathcal{T}^{>}\}$ .

In order to ensure termination of Algorithm 4.1, we assign a “depth” to each toric datum. The initial toric datum given as the input of Algorithm 4.1 has depth 0. We further let  $\text{BALANCE}(\mathcal{T})$  and  $\text{SIMPLIFY}(\mathcal{T})$  return toric data of the same depth as  $\mathcal{T}$ . If, having chosen  $(i, j, t_i, t_j)$  as part of the reduction step, we have  $w(\mathcal{T}') > w(\mathcal{T})$  for some  $\mathcal{T}' \in \mathfrak{C}'(i, j, t_i, t_j)$ , then we increase the depth of  $\mathcal{T}'$ . Termination is guaranteed by letting  $\text{REDUCE}(\mathcal{T})$  fail whenever the depth of  $\mathcal{T}$  exceeds some constant value. While this approach is less elegant than a strictly greedy approach, where we would e.g. insist that  $w(\mathcal{T}') < w(\mathcal{T})$  for all  $\mathcal{T}' \in \mathfrak{C}'(i, j, t_i, t_j)$ , it is more powerful in practice.

For an illustration of the reduction step applied to a “real-life” example, see §9.1.



## 8 Practical matters

### 8.1 Introducing “Zeta”

The Python-package `Zeta` [23] for Sage [26] provides an implementation of Algorithm 4.1 for computing topological subalgebra, ideal, and submodule zeta functions for  $k = \mathbf{Q}$ . Sage natively supports computations with rational polyhedra and we use these capabilities to simulate computations with half-open cones, see §8.4. Polynomial arithmetic and Gröbner bases computations are handled by `SINGULAR` [14]. For the computations of mixed volumes mentioned in §6.6, we use `Gfan` [18]. While Sage does provide functionality for computing triangulations, we use the fast implementation provided by `Normaliz` [5] if it is available. In order to use `Zeta` to compute topological zeta functions associated with nilpotent groups via Theorem 2.3, one may use the `GAP`-package `Guarana` [3, 27] which provides an effective version of the Mal’cev correspondence.

### 8.2 On the scope of Algorithm 4.1 and its implementation

**Theoretical limitations.** While Algorithm 4.1 allows us to compute far more topological zeta functions than [22, Thm 6.7] alone could, it is fairly easy to produce examples that seem completely resistant to our approach. For example, while the vast majority of known topological and local subalgebra and ideal zeta functions arise from nilpotent Lie rings, to the author’s knowledge, not a single example of any such zeta function associated with a nilpotent Lie ring of class  $\geq 5$  has ever been computed. In particular, there are various examples of nilpotent Lie rings of additive rank 6 whose topological and local subring and ideal zeta functions remain unknown—our method has so far been unable to remedy this. As the additive rank of the non-associative ring under consideration increases or the assumption that it be nilpotent and Lie is relaxed, examples amenable to our method become rare.

**Practical issues.** So far, the most successful applications of `Zeta` were concerned with (nilpotent) associative, commutative, or Lie rings of additive rank at most 6. Even in the case of nilpotent Lie rings of rank 6, some of the computations carried out by the author took several months to complete (using 16 parallel processes on an ordinary computer, see §8.3). In such cases, the most expensive step in Algorithm 4.1 is the final line. At this point, `regular` will be populated with possibly thousands of regular toric data. For each  $(\mathcal{D}_0; g_1, \dots, g_e) \in \text{regular}$ , we then consider each of the half-open cones  $\mathcal{D}_0^J$  indexed by  $J \subset \{1, \dots, e\}$  yielding perhaps tens of thousands of half-open cones in total. Finally, the triangulation step in `EVALUATETOPOLOGICALLY` will often decompose each  $\bar{\mathcal{D}}_0^J$  into possibly tens or even hundreds of thousands of simplicial cones; note that for examples of rank 6, the ambient Euclidean space of each  $\mathcal{D}_0^J$  will have dimension at least  $21 = \frac{6 \cdot 7}{2}$ .

### 8.3 EVALUATETOPOLOGICALLY in practice

As mentioned in Remark 6.15(i), in our implementation of `EVALUATETOPOLOGICALLY` we immediately apply specialisations of the form  $\mathbf{s}_j \mapsto \mathbf{s} - j$  (see Remark 3.4) needed to

recover the desired univariate topological zeta function. Moreover, in order to avoid costly rational function arithmetic, we do not actually carry out either the summation in line 14 of Algorithm 14 nor that in the proof of Lemma 6.11. Instead, we first compute the final output of Algorithm 4.1,  $Z_{\text{top}}(\mathbf{s}) \in \mathbf{Q}(\mathbf{s})$  say, as an unevaluated (possibly large) sum of rational functions of the form  $\frac{c}{(a_1 s - b_1) \cdots (a_d s - b_d)}$  for suitable integers  $a_i, b_i, c$  (arising from simplicial cones in Lemma 6.11 and Euler characteristics in §6.6). As we construct these rational functions, we keep track of a “candidate denominator” of  $Z_{\text{top}}(\mathbf{s})$ , i.e. a polynomial  $g \in \mathbf{Z}[\mathbf{s}]$ ,  $g \neq 0$  with  $gZ_{\text{top}}(\mathbf{s}) \in \mathbf{Z}[\mathbf{s}]$ . Using Lemma 6.5, we may then recover  $Z_{\text{top}}(\mathbf{s})$  using random evaluation and polynomial interpolation.

After successful termination of the main loop (lines 1–13) in Algorithm 4.1, the remaining tasks of computing Euler characteristics, triangulating cones, and evaluating rational functions can be trivially parallelised and our implementation makes use of this.

## 8.4 Computing with half-open cones

We defined our basic data structure, the toric data from §3, in terms of rational half-open cones  $\mathcal{C}_0 \subset \mathbf{R}_{\geq 0}^n$  since they are the smallest collection of subsets of  $\mathbf{R}^n$  which contains  $\mathbf{R}_{\geq 0}^n$  and which is stable under the effects of BALANCE, SIMPLIFY, and REDUCE. However, half-open cones and polyhedra (rational or not) are scarcely used in the literature and they are usually not directly supported by existing software. Apart from triangulating closed rational cones, the only computational tasks involving half-open cones that we actually relied upon are the following:

- (i) Compute the intersection of two rational half-open cones.
- (ii) Decide if a rational half-open cone is empty.
- (iii) Construct the closure of a non-empty rational half-open cone.
- (iv) Decide membership of a vector in the dual of a rational half-open cone.

As  $\mathcal{C}_0^* = \bar{\mathcal{C}}_0^*$  for a half-open cone  $\mathcal{C}_0$ , (iii) reduces (iv) to the closed case (which is standard).

**Definition 8.1.** By a (polyhedral) **model** of a rational half-open cone  $\mathcal{C}_0 \subset \mathbf{R}^n$ , we mean a rational polyhedron  $\mathcal{P}_0 \subset \mathbf{R}^n$  such that  $\mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{P}_0 \cap \mathbf{Z}^n$  and  $\mathbf{N}\mathcal{P}_0 = \mathcal{P}_0$ .

For example, for each  $a \in \mathbf{N}$ , the closed interval  $[1/a, \infty)$  is a model of the open interval  $(0, \infty)$ . As we will now explain, we may replace half-open cones by models in our computations.

**Proposition 8.2.** *Let  $\mathcal{C}_0, \mathcal{C}'_0 \subset \mathbf{R}^n$  be rational half-open cones.*

- (i)  $\mathcal{C}_0$  admits a model.
- (ii) Let  $\mathcal{P}_0 \subset \mathcal{C}_0$  and  $\mathcal{P}'_0 \subset \mathcal{C}'_0$  be models. Then  $\mathcal{P}_0 \cap \mathcal{P}'_0$  is a model of  $\mathcal{C}_0 \cap \mathcal{C}'_0$ .
- (iii) Let  $\mathcal{P}_0$  be a model of  $\mathcal{C}_0$ . Then  $\mathcal{C}_0$  is empty if and only if  $\mathcal{P}_0$  is empty.
- (iv) If  $\mathcal{C}_0 \neq \emptyset$  and  $\mathcal{P}_0$  is a model of  $\mathcal{C}_0$ , then  $\bar{\mathcal{C}}_0$  is the smallest cone containing  $\mathcal{P}_0$ .

(v) If  $\mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{C}'_0 \cap \mathbf{Z}^n$ , then  $\mathcal{C}_0 = \mathcal{C}'_0$ . Hence,  $\mathcal{C}_0$  is determined by any of its models.

*Proof.*

- (i) There are finitely many  $\phi_i, \chi_j \in \mathbf{Z}^n$  ( $i \in I, j \in J$ ) with  $\mathcal{C}_0 = \{\omega \in \mathbf{R}^n : \forall i \in I. \langle \phi_i, \omega \rangle \geq 0, \forall j \in J. \langle \chi_j, \omega \rangle > 0\}$ . Hence,  $\{\omega \in \mathbf{R}^n : \forall i \in I. \langle \phi_i, \omega \rangle \geq 0, \forall j \in J. \langle \chi_j, \omega \rangle \geq 1\}$  is a model of  $\mathcal{C}_0$ .
- (ii) Obvious.
- (iii) The relative interior  $\text{relint}(\mathcal{C})$  of a non-empty rational cone  $\mathcal{C} \subset \mathbf{R}^n$  satisfies  $\text{relint}(\mathcal{C}) \cap \mathbf{Q}^n \neq \emptyset$  (in fact,  $\text{relint}(\mathcal{C}) \cap \mathbf{Q}^n$  is dense in  $\mathcal{C}$ ) and hence  $\text{relint}(\mathcal{C}) \cap \mathbf{Z}^n \neq \emptyset$ . Hence, if  $\mathcal{P}_0 = \emptyset$ , then  $\mathcal{C}_0 = \emptyset$ . Suppose that  $\mathcal{P}_0 \neq \emptyset$ . There exists  $\beta \in \mathcal{P}_0 \cap \mathbf{Q}^n$  and thus  $a\beta \in \mathcal{P}_0 \cap \mathbf{Z}^n = \mathcal{C}_0 \cap \mathbf{Z}^n$  for some  $a \in \mathbf{N}$  whence  $\mathcal{C}_0 \neq \emptyset$ .
- (iv) For  $\omega \in \mathcal{C}_0 \cap \mathbf{Q}^n$ , there exists  $a \in \mathbf{N}$  with  $a\omega \in \mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{P}_0 \cap \mathbf{Z}^n \subset \mathcal{P}_0$ . Hence, every cone containing  $\mathcal{P}_0$  also contains  $\mathcal{C}_0 \cap \mathbf{Q}^n$ , a dense subset of  $\bar{\mathcal{C}}_0$ . Similarly,  $\mathcal{P}_0 \cap \mathbf{Q}^n \subset \bar{\mathcal{C}}_0$  so that  $\mathcal{P}_0 \subset \bar{\mathcal{C}}_0$ .
- (v) By (iii), we may assume that  $\mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{C}'_0 \cap \mathbf{Z}^n \neq \emptyset$ . Let  $\mathcal{P}_0$  be a model of  $\mathcal{C}_0$ . Then  $\mathcal{P}_0$  is also a model of both  $\mathcal{C}'_0$  and  $\mathcal{C}''_0 := \mathcal{C}_0 \cap \mathcal{C}'_0$ . By (iv),  $\mathcal{C}_0, \mathcal{C}'_0, \mathcal{C}''_0$  all have the same closure,  $\mathcal{C}$  say. Suppose that  $\mathcal{C}''_0 \neq \mathcal{C}_0$ . Then there exists a face  $\tau$  of  $\mathcal{C}$  such that  $\text{relint}(\tau) \subset \mathcal{C}_0$  but  $\tau \cap \mathcal{C}''_0 = \emptyset$ . Let  $\omega \in \text{relint}(\tau) \cap \mathbf{Z}^n$ . Then  $\omega \in \mathcal{C}_0 \cap \mathbf{Z}^n = \mathcal{C}''_0 \cap \mathbf{Z}^n$ , a contradiction. Hence,  $\mathcal{C}_0 \subset \mathcal{C}'_0$  and so  $\mathcal{C}_0 = \mathcal{C}'_0$  by symmetry.  $\blacklozenge$

Given a model  $\mathcal{P}_0$  of  $\mathcal{C}_0$ , it remains to recover  $\bar{\mathcal{C}}_0$  explicitly from  $\mathcal{P}_0$ .

**Lemma 8.3.** *Let  $\mathcal{Q} = \text{conv}(\alpha_1, \dots, \alpha_e) \subset \mathbf{R}^n$  be a non-empty polytope and  $\mathcal{D} \subset \mathbf{R}^n$  be a cone. Write  $\mathcal{C} = \text{cone}(\alpha_1, \dots, \alpha_e)$ . Then  $\mathcal{C} + \mathcal{D}$  is the smallest cone containing  $\mathcal{Q} + \mathcal{D}$ .*

*Proof.* Clearly,  $\mathcal{C} + \mathcal{D}$  is a cone containing  $\mathcal{P} := \mathcal{Q} + \mathcal{D}$ . Let  $\mathcal{B} \supset \mathcal{P}$  be a cone. Then  $\mathcal{Q} \subset \mathcal{B}$  and hence  $\mathcal{C} \subset \mathcal{B}$ . Fix  $\mathbf{x} \in \mathcal{Q}$  and let  $\mathbf{y} \in \mathcal{D}$ . Then  $\mathbf{x} + a\mathbf{y} \in \mathcal{P}$  for  $a > 0$  whence  $a^{-1}\mathbf{x} + \mathbf{y} \in \mathcal{B}$ . As  $\mathcal{B}$  is closed, we conclude that  $\mathbf{y} \in \mathcal{B}$  and thus  $\mathcal{D} \subset \mathcal{B}$ .  $\blacklozenge$

Recall that (rational) polyhedra in  $\mathbf{R}^n$  are exactly the sets of the form  $\mathcal{Q} + \mathcal{D}$ , where  $\mathcal{Q} \subset \mathbf{R}^n$  is a (rational) polytope and  $\mathcal{D} \subset \mathbf{R}^n$  is a (rational) cone, see e.g. [4, Thm 4.13]. Let  $\mathcal{P}_0$  be a model of a non-empty rational half-open cone  $\mathcal{C}_0$ . Writing  $\mathcal{P}_0 = \mathcal{Q} + \mathcal{D}$  for (rational)  $\mathcal{Q}, \mathcal{D}$  as in Lemma 8.3, Proposition 8.2(iv) allows us to recover  $\bar{\mathcal{C}}_0$ .

## 9 Examples

We discuss examples of previously unknown topological zeta functions computed using Zeta [23]. For more examples, we refer to the database of topological subring, ideal, and submodule zeta functions included with Zeta.

## 9.1 Five-dimensional nilpotent Lie algebras: $\text{Fil}_4$

As in [30, Thm 3.6], let  $\text{Fil}_4$  be the nilpotent Lie ring with  $\mathbf{Z}$ -basis  $(e_1, \dots, e_5)$  and Lie bracket  $[e_1, e_2] = e_3$ ,  $[e_1, e_3] = e_4$ ,  $[e_1, e_4] = e_5$ ,  $[e_2, e_3] = e_5$ , and  $[e_i, e_j] = 0$  for  $i \leq j$  not listed above. As we explained in [22, §7.3], with the sole exception of  $\text{Fil}_4 \otimes_{\mathbf{Z}} \mathbf{C}$ , each of the 16 isomorphism classes of non-trivial nilpotent Lie  $\mathbf{C}$ -algebras of dimension at most 5 admits a  $\mathbf{Z}$ -form whose local subring zeta functions have been computed. We can use Algorithm 4.1 and **Zeta** to confirm that for the 15 known types, the topological zeta function coincides with the one naively deduced from  $p$ -adic formulae. The local subring zeta functions of  $\text{Fil}_4$  have so far resisted attempts at computing them [30, p. 57]. In [22, Eqn (7.8)], we announced that

$$\begin{aligned} \zeta_{\text{Fil}_4, \text{top}}(s) = & (392031360s^9 - 5741480808s^8 + 37286908278s^7 - 140917681751s^6 + \\ & 341501393670s^5 - 550262853249s^4 + 589429290044s^3 - \\ & 404678115300s^2 + 161557332768s - \\ & 28569052512) / (3(15s - 26)(7s - 12)(7s - 13)(6s - 11)^3 \\ & (5s - 8)(5s - 9)(4s - 7)^2(3s - 4)(2s - 3)(s - 1)s). \end{aligned} \quad (9.1)$$

For a group-theoretic interpretation, since the topological subgroup zeta function of a torsion-free, finitely generated nilpotent group  $G$  only depends on the  $\mathbf{C}$ -isomorphism type of  $\mathfrak{L}(G) \otimes_{\mathbf{Q}} \mathbf{C}$  (see [22, Prop. 5.19(ii)]), we thus obtain a complete classification of topological subgroup zeta functions of nilpotent groups of Hirsch length at most 5, see the database included with **Zeta**.

We will now provide details on the computation leading to (9.1). In doing so, we illustrate the key steps of Algorithm 4.1.

**Constructing an initial toric datum.** The first step is to construct Laurent polynomials as in Theorem 2.2(i) (see the end of §2) and an associated toric datum as in Remark 3.4; the input of Algorithm 4.1 then consists of said toric datum,  $\mathcal{T}^0$  say, and a  $(0, 1)$ -matrix  $\beta \in M_{5 \times 15}(\mathbf{N}_0)$  (which can be easily constructed as in Remark 3.4).

Using the defining basis  $(e_1, \dots, e_5)$  of  $\text{Fil}_4$  and after performing simplification steps (see §7.2), we thus obtain the initial toric datum  $\mathcal{T}^0 = (\mathcal{C}_0; f_1, f_2, f_3)$ , where

$$\mathcal{C}_0 = \left\{ (\omega_1, \dots, \omega_{15}) \in \mathbf{R}_{\geq 0}^{15} : \begin{aligned} & \omega_{10} \leq \omega_1 + \omega_6, \quad \omega_{13} \leq \omega_1 + \omega_{10}, \\ & \omega_{15} \leq \omega_6 + \omega_{10}, \quad \omega_{15} \leq \omega_1 + \omega_{13} \end{aligned} \right\}$$

and  $f_1, f_2, f_3 \in \mathbf{Q}[X_1^{\pm 1}, \dots, X_{15}^{\pm 1}]$  are given by

$$\begin{aligned} f_1 &= \underline{X_2 X_{10} X_{15}^{-1}} - X_1 X_{10} X_{13}^{-1} X_{14} X_{15}^{-1} + X_1 X_{11} X_{15}^{-1} \\ f_2 &= \underline{X_1 X_7 X_{13}^{-1}} - X_1 X_6 X_{10}^{-1} X_{11} X_{13}^{-1} \\ f_3 &= \underline{X_1 X_7 X_{13}^{-1} X_{14} X_{15}^{-1}} - X_2 X_7 X_{15}^{-1} + X_3 X_6 X_{15}^{-1} - X_1 X_8 X_{15}^{-1} + \\ & \quad X_1 X_6 X_{10}^{-1} X_{12} X_{15}^{-1} - X_1 X_6 X_{10}^{-1} X_{11} X_{13}^{-1} X_{14} X_{15}^{-1}; \end{aligned}$$

please ignore the underlines at first reading.

**Balancing and regularity testing.** Our next task is to decompose  $\mathcal{C}_0$  using the normal cones of  $\mathcal{N} := \text{New}(f_1 f_2 f_3) \subset \mathbf{R}^{15}$  to obtain a system of *balanced* toric data. Using Sage, we find that  $\mathcal{N}$  is a 6-dimensional polytope with 27-vertices and a total of 395 faces; note that Sage regards  $\emptyset$  as a face of  $\mathcal{N}$  but that we do not. For each face  $\tau \subset \mathcal{N}$ , we then simplify the balanced toric datum  $(\mathcal{C}_0 \cap N_\tau(\mathcal{N}); f_1, f_2, f_3)$  (see §7.2) and test it for regularity (see §5.3). It turns out that all but 4 of these 395 toric data are already regular.

**Singularity and reduction.** We now consider one of the aforementioned four singular toric data arising from  $(\mathcal{C}_0; f_1, f_2, f_3)$  in detail, namely  $(\tilde{\mathcal{C}}_0; f_1, f_3)$ , where

$$\begin{aligned} \tilde{\mathcal{C}}_0 = \Big\{ (\omega_1, \dots, \omega_{15}) \in \mathbf{R}_{\geq 0}^{15} : & \omega_1 \leq \omega_2 + \omega_{13}, & \omega_{10} \leq \omega_1 + \omega_6, \\ & \omega_{13} \leq \omega_1 + \omega_7, & \omega_{13} \leq \omega_1 + \omega_{10}, \\ & \omega_{15} \leq \omega_1 + \omega_{13}, & \omega_{15} \leq \omega_6 + \omega_{10}, \\ & \omega_1 + \omega_{14} = \omega_2 + \omega_{13}, \\ & \omega_2 + \omega_7 < \omega_1 + \omega_8, & \omega_2 + \omega_7 < \omega_3 + \omega_6, \\ & \omega_2 + \omega_{10} < \omega_1 + \omega_{11}, & \omega_7 + \omega_{10} < \omega_6 + \omega_{11}, \\ & \omega_2 + \omega_7 + \omega_{10} < \omega_1 + \omega_6 + \omega_{12} \Big\}, \end{aligned}$$

which is a 14-dimensional half-open cone contained in  $\mathcal{C}_0$ . We see that the initial forms of  $f_1$ ,  $f_2$ , and  $f_3$  on  $\tilde{\mathcal{C}}_0$  are exactly the underlined parts from above. In particular, since  $\text{in}_{\tilde{\mathcal{C}}_0}(f_2)$  is a term, we can see why  $f_2$  has been discarded by the simplification step.

One checks that while  $\text{in}_{\tilde{\mathcal{C}}_0}(f_1) = 0$  and  $\text{in}_{\tilde{\mathcal{C}}_0}(f_3) = 0$  both define smooth hypersurfaces within  $\mathbf{T}_{\mathbf{Q}}^{15}$ , the rank condition defining regularity is violated precisely on the subvariety (subtorus, in fact) defined by  $X_1 X_{14} = X_2 X_{13}$ . Indeed, looking at the initial forms of  $f_1$  and  $f_3$ , we see that the failure of regularity is due to these initial forms being *identical* up to multiplication by a unit in  $\mathbf{Q}[X_1^{\pm 1}, \dots, X_{15}^{\pm 1}]$ . The reduction procedure explained in §7.3 arose from the observation that such geometrically simple causes of singularity are remarkably common in practice.

Reduction can “repair” the failure of regularity of  $(\tilde{\mathcal{C}}_0; f_1, f_3)$  as follows. Define terms  $t_1 := X_2 X_{10} X_{15}^{-1}$  and  $t_3 := -X_2 X_7 X_{15}^{-1}$  of  $f_1$  and  $f_3$ , respectively; note that  $t_1^{-1} t_3 = -X_7 X_{10}^{-1}$ . Define  $\tilde{\mathcal{C}}_0^{\leq} = \{\omega \in \tilde{\mathcal{C}}_0 : \omega_{10} \leq \omega_7\}$ ,  $\tilde{\mathcal{C}}_0^> = \{\omega \in \tilde{\mathcal{C}}_0 : \omega_7 < \omega_{10}\}$ ,  $g_3 = f_3 - t_1^{-1} t_3 f_1$ , and  $g_1 = f_1 - t_1 t_3^{-1} f_3$ . Then

$$\begin{aligned} g_3 &= X_3 X_6 X_{15}^{-1} - X_1 X_8 X_{15}^{-1} + X_1 X_7 X_{10}^{-1} X_{11} X_{15}^{-1} + \\ &\quad X_1 X_6 X_{10}^{-1} X_{12} X_{15}^{-1} - X_1 X_6 X_{10}^{-1} X_{11} X_{13}^{-1} X_{14} X_{15}^{-1} \text{ and} \\ g_1 &= X_3 X_6 X_7^{-1} X_{10} X_{15}^{-1} - X_1 X_7^{-1} X_8 X_{10} X_{15}^{-1} + X_1 X_{11} X_{15}^{-1} + \\ &\quad X_1 X_6 X_7^{-1} X_{12} X_{15}^{-1} - X_1 X_6 X_7^{-1} X_{11} X_{13}^{-1} X_{14} X_{15}^{-1}. \end{aligned}$$

Setting  $\mathcal{T}^{\leq} := (\tilde{\mathcal{C}}_0^{\leq}; f_1, g_3)$  and  $\mathcal{T}^> := (\tilde{\mathcal{C}}_0^>; g_1, f_3)$ , we obtain a partition  $\{\mathcal{T}^{\leq}, \mathcal{T}^>\}$  of  $(\tilde{\mathcal{C}}_0; f_1, f_3)$ . As we will now explain, our particular choice of a reduction candidate

(terminology as in §7.3) eliminates the source of the singularity of  $(\tilde{\mathcal{C}}_0; f_1, f_3)$  that we isolated above. In general, it is possible for choices of reduction candidates to introduce new singularities. In order to verify that this is not the case here, we apply the balancing procedure from §5.2 followed by simplification to both  $\mathcal{T}^{\leq}$  and to  $\mathcal{T}^>$ . In doing so, each of these toric data is partitioned into 31 balanced conditions, 31 being the number of faces of the Newton polytope of  $g_1$  and of  $g_3$ . Fortunately, every single one the resulting 62 toric data is now regular, which concludes our efforts regarding  $(\tilde{\mathcal{C}}_0; f_1, f_3)$ .

The other three singular toric data mentioned above can be handled in a very similar way. Each of them is cut in two by reduction and each piece is then decomposed into 15 regular conditions by BALANCE. In particular, in each case, a single application of the reduction step followed by balancing and simplification immediately yields regular conditions only—this is not to be expected in general.

After completion of the main loop in Algorithm 4.1, we have constructed a total of 543 regular toric data constituting a partition of  $\mathcal{T}^0$  from above.

**Final stage.** It remains to apply the function EVALUATETOPOLOGICALLY to each of the 543 aforementioned toric data and to recover  $\zeta_{\text{Fil}_4, \text{top}}(\mathbf{s})$  from a sum of rational functions. As it is unlikely to offer any new insights, we chose not to give details on the tedious acts of computing Euler characteristics, triangulating cones, and manipulating rational functions that constitute this step.

**Stats.** We briefly indicate the extent to which practical applications of Algorithm 4.1 rely on machine computations. Thus, using Zeta on an Intel Xeon E5-2670 (8 cores) running Sage 6.3, the computation of  $\zeta_{\text{Fil}_4, \text{top}}(\mathbf{s})$  sketched above took about 97 minutes in total. The main loop in Algorithm 4.1 was completed after 2 minutes; the vast majority of time was then spent in the final line of Algorithm 4.1 which used 16 parallel processes, see §8.3. Using polynomial interpolation, the final formula (9.1) was then recovered from a sum of 12,869,940 rational functions as explained in §8.3.

## 9.2 Other examples

There are various interesting examples which are similar to  $\text{Fil}_4$  in the sense that a single application of the reduction step to singular toric data already suffices. One such example is given by the topological submodule zeta function of the full unipotent group  $U_5(\mathbf{Z}) \leq \text{GL}_5(\mathbf{Z})$  acting on its natural module discussed in [22, §7.3]. For a non-nilpotent, commutative, and associative example, we find the topological subring zeta function of  $\mathbf{Z}[X]/X^4$  to be

$$\zeta_{\mathbf{Z}[X]/X^4, \text{top}}(\mathbf{s}) = \frac{2021760s^5 - 8509620s^4 + 14322332s^3 - 12036071s^2 + 5044460s - 842400}{168480(6s - 5)(4s - 3)(s - 1)^6s}.$$

A computation of similar overall complexity which however requires multiple iterations of reduction yields the formula for  $\zeta_{\text{gl}_2(\mathbf{Z}), \text{top}}(\mathbf{s})$  announced in [22, (7.7)]. For a more complicated example, consider  $\text{Fil}_4 \oplus (\mathbf{Z}, 0)$ , where  $(\mathbf{Z}, 0)$  denotes  $\mathbf{Z}$  regarded as an abelian

Lie ring. After about 4 days (same machine as for  $\text{Fil}_4$  above), **Zeta** reports that

$$\begin{aligned} \zeta_{\text{Fil}_4 \oplus (\mathbf{Z}, 0), \text{top}}(s) = & (52839554826240s^{15} - 1612571385729024s^{14} + \\ & 22945067840268288s^{13} - 201917310138409536s^{12} + \\ & 1228942670032455984s^{11} - 5479610770178424720s^{10} + \\ & 18489925054934205732s^9 - 48077179247205683304s^8 + \\ & 97118269735864324559s^7 - 152405042677332499112s^6 + \\ & 184268407184801648476s^5 - 168562287295854189878s^4 + \\ & 112921211241642321545s^3 - 52295417007047312650s^2 + \\ & 14969814525806597400s - 1996549752637440000) / \\ & (48(15s - 31)(15s - 34)(13s - 28)(12s - 25)(9s - 20)(7s - 15) \\ & (7s - 16)(6s - 13)^3(5s - 11)(4s - 9)^3(3s - 5)(s - 1)(s - 2)^4s), \quad (9.2) \end{aligned}$$

For further examples of topological zeta functions of the type considered in this article, see **Zeta** and the database that comes with it.

### 9.3 On the reliability of our computations

When it comes to trusting computer output such as the examples given above, caution is certainly warranted. Apart from possible bugs in the author’s code, the sheer number of mathematical libraries and programs relied upon by **Zeta** is a natural source of concern.

**Independent confirmation.** As a simple test, we can use the many examples of local zeta functions computed by Woodward and others and compare the associated “naive” topological zeta functions (obtained via symbolic expansion in  $p - 1$  as indicated in the introduction) with the ones obtain using **Zeta**, assuming our method applies. Our implementation passes this test for all examples from [13] that we considered. Conversely, our machine computations thus provide evidence for the correctness of these formulae which were often obtained using complicated, at least partially manual, and often undocumented computations.

**Conjectures.** For genuinely new examples such as the topological subring zeta functions of  $\text{Fil}_4$  and  $\text{Fil}_4 \oplus (\mathbf{Z}, 0)$ , we regard the peculiar conjectural features of topological zeta functions from [22, §8] as further evidence of the reliability of our implementation—indeed, computational errors can easily destroy these properties. The lengthy formula (9.2), for example, has all the properties predicted by the conjectures in [22, §8]. In addition, we observe that  $\zeta_{\text{Fil}_4, \text{top}}(s)$  and  $\zeta_{\text{Fil}_4 \oplus (\mathbf{Z}, 0), \text{top}}(s)$  “agree at infinity” in the following sense. Given any non-associative ring  $\mathcal{A}$  of additive rank  $d$ , let  $m(\mathcal{A}) := \zeta_{\mathcal{A}, \text{top}}(s^{-1})s^d|_{s=0}$ . The “degree conjecture” [22, Conj. I] asserts that  $0 \neq m(\mathcal{A}) \neq \infty$ .

**Conjecture V.**  $m(\mathcal{A}) = m(\mathcal{A} \oplus (\mathbf{Z}, 0))$ .

For instance, (9.1) and (9.2) show that  $m(\text{Fil}_4) = m(\text{Fil}_4 \oplus (\mathbf{Z}, 0)) = 463/1350$ .

The effect of the operation  $\mathcal{A} \mapsto \mathcal{A} \oplus (\mathbf{Z}, 0)$  (let alone arbitrary direct sums) on subring or ideal zeta functions (local or topological) is poorly understood in general. For a few specific examples of rings  $\mathcal{A}$ , formulae for local zeta functions of  $\mathcal{A} \oplus (\mathbf{Z}^r, 0)$  are known for all  $r \geq 0$ , see [13]; these formulae are consistent with Conjecture V. The simple patterns exhibited by the formulae for known instances of such families seem to be exceptional, as e.g. suggested by various examples of topological zeta functions included with Zeta. The experimental evidence underpinning Conjecture V is all the more remarkable in view of the generally increased complexity of  $\zeta_{\mathcal{A} \oplus (\mathbf{Z}, 0), \text{top}}(\mathbf{s})$  compared with  $\zeta_{\mathcal{A}, \text{top}}(\mathbf{s})$ .

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